H-space and H-cospace
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H-space and H-cospace

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ABSTRACT

An $H$-space is a pointed topological space $K$ equipped with a multiplication map $\mu : K \times K \to K$ which induces a monoid structure on the set $[X, K]$ for every pointed space $X$. An $H$-cospace is pointed topological space which is dual to an $H$-space. The most important example of an $H$-space (resp. $H$-cospace) is the loop space (resp. the suspension) of a pointed space. The sphere $S^n$ is homeomorphic to the suspension of the sphere $S^{n-1}$ (Lemma 3.6), so it is an $H$-cogroup. There is a natural isomorphism $[SX, Y] \cong [X, \Omega Y]$, which means that $\pi_n(X) \cong \pi_{n-1}(\Omega X)$. 
국 문 초 록

H-space는 monoid 구조를 갖는 곱 사상으로 이루어진 위상 공간이고 H-cospace는 H-space가 생성으로 이루어진 위상 공간을 말한다. H-space의 가장 중요한 예는 loop space이며 H-cospace의 가장 중요한 예는 suspension이다. Lemma 3.6에 의해 $S^n$과 $S^{n-1}$의 suspension이 동형이며 따라서 이는 H-cospace가 된다. 자연스러운 동형사상 $[SX, Y] \simeq [X, \Omega Y]$이 있고, 이는 $\pi_n(X) \simeq \pi_{n-1}(\Omega X)$를 의미한다.
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Chapter 1. Introduction

The homotopy theory was first introduced by Henry Poincare about one hundred years ago and, since then, has been the main ingredient of algebraic topology. It was actually the start point of topology which is the most remarkable new area in the 20th-century mathematics.

$H$-spaces and $H$-cospaces are very important spaces in the basic homotopy theory. An $H$-space (resp. $H$-group) is a pointed topological space $K$ equipped with a multiplication map $\mu : K \times K \to K$ which induces a monoid (resp. group) structure on the set $[X, K]$ for every pointed space $X$, where $[X, K]$ denotes the set of homotopy classes of base point preserving maps from $X$ to $K$. A topological group is, of course, an $H$-group. In fact, an $H$-group is a homotopic version of topological group. An important example of an $H$-space is the loop space, that is, for every pointed space $X$, the loop space at $x_0$

$$\Omega(X, x_0) = \{ w : I \to X \mid w(0) = w(1) = x_0 \},$$

is an $H$-space (Theorem 3.3).

An $H$-cospace is pointed topological space which is dual to an $H$-space. A pointed space $K$ is called an $H$-cospace (resp. $H$-cogroup) if it has a comultiplication map $\mu' : K \to K \vee K$ so
that \([X,K]\) forms a structure of a monoid (resp. group) for every pointed space \(X\). The most important example of an \(H\)-cogroup is the suspension of a space (Theorem 3.5). The sphere \(S^n\) is homeomorphic to the suspension of the sphere \(S^{n-1}\) (Lemma 3.6), so is an \(H\)-cogroup. Hence for every pointed space \(X\), \([S^n,X]\) forms a group, and is called the \(n\)-th homotopy group which plays a very important role in the homotopy theory.

The loop space functor and the suspension functor are adjoint, that is, we have the natural isomorphism \([SX,Y] \approx [X,\Omega Y]\), which means that, in particular, \(\pi_n(X) \cong \pi_{n-1}(\Omega X)\).
Chapter 2. $H$-spaces and $H$-cospaces

Definition 2.1 If $X$ and $Y$ are spaces and if $f_0$, $f_1$ are continuous maps from $X$ to $Y$ then $f_0$ is homotopic to $f_1$, denoted by $f_0 \simeq f_1$, if there is a continuous map $F : X \times I \rightarrow Y$ with $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. Such a map $F$ is called a homotopy.

Theorem 2.2 Homotopy is an equivalence relation on the set of all continuous maps $X \rightarrow Y$.

Proof. Reflexivity. If $f : X \rightarrow Y$, define $F : X \times I \rightarrow Y$ by $F(x, t) = f(x)$ for all $t \in I$. Clearly, $F : f \simeq f$.

Symmetry. Assume that $f \simeq g$, so there is a continuous map $F : X \times I \rightarrow Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Define $G : X \times I \rightarrow Y$ by $G(x, t) = F(x, 1 - t)$. Then $G : g \simeq f$.

Transitivity. Assume that $F : f \simeq g$ and $G : g \simeq h$. Define $H : X \times I \rightarrow Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$
Then $H : f \simeq g$. 

**Definition 2.3** If $f : X \to Y$ is continuous, its **homotopy class** is the equivalence class

$$[f] = \{ g : X \to Y \mid g \simeq f \}.$$ 

The family of such homotopy classes is denoted by $[X, Y]$. 

**Definition 2.4** A **category** $\mathcal{C}$ consists of three ingredients: a class of objects, $\text{obj} \mathcal{C}$; sets of morphism $\text{Hom}(A, B)$, one for every ordered pair $A, B \in \text{obj} \mathcal{C}$; composition $\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$ denoted by $(f, g) \mapsto g \circ f$ for every $A, B, C \in \text{obj} \mathcal{C}$, satisfying the following axioms:

1. composition is associative when defined;
2. for each $A \in \text{obj} \mathcal{C}$ there exists an identity $1_A \in \text{Hom}(A, A)$ satisfying $1_A \circ f = f$ for every $f \in \text{Hom}(B, A)$ all $B \in \text{obj} \mathcal{C}$, and $g \circ 1_A = g$ for every $g \in \text{Hom}(A, C)$, all $C \in \text{obj} \mathcal{C}$.

**Definition 2.5** $T^2 = \text{Top}^2$. Here $\text{obj} T^2$ consists of all ordered pairs $(X, A)$, where $X$ is a topological space and $A$ is a subspace of $X$. A morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subseteq B$. $\text{Top}^2$ is called the category of pairs of topological spaces.
Definition 2.6 $T = \text{Top}^*$. Here $\text{obj } T$ consists of all ordered pairs $(X, x_0)$ where $X$ is a topological space and $x_0$ is a point of $X$. $\text{Top}^*$ is a subcategory of $\text{Top}^2$ (subspaces here are always one-point subspaces), and it is called the category of pointed spaces: $x_0$ is called the base point of $(X, x_0)$, and morphisms are called pointed maps (or base point preserving maps).

Definition 2.7 An $H$-space is pointed space $(K, k_0)$ with multiplication map $\mu : K \times K \to K$ such that for the constant map $k_0$, the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{(k_0, 1)} & K \times K \\
\downarrow 1 & & \downarrow \mu \\
K & \xrightarrow{1} & K
\end{array}
\]

commutes up to homotopy: $\mu \circ (1, k_0) \simeq 1 \simeq \mu \circ (k_0, 1)$. We say $\mu$ is homotopy associative if the diagram

\[
\begin{array}{ccc}
K \times K \times K & \xrightarrow{\mu \times 1} & K \times K \\
\downarrow 1 \times \mu & & \downarrow \mu \\
K \times K & \xrightarrow{\mu} & K
\end{array}
\]

commutes up to homotopy: $\mu \circ (\mu \times 1) \simeq \mu \simeq \mu \circ (1 \times \mu)$. A
map $\nu : (K, k_0) \to (K, k_0)$ is called a homotopy inverse if the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{(\nu, 1)} & K \times K \\
\downarrow & & \downarrow \mu \\
k_0 & \searrow & k_0
\end{array}
$$

commutes up to homotopy: $\mu \circ (\nu, 1) \simeq k_0 \simeq \mu \circ (1, \nu)$. We say $\mu$ is homotopy commutative if the diagram

$$
\begin{array}{ccc}
K \times K & \xrightarrow{T} & K \times K \\
\downarrow & & \downarrow \mu \\
K & \searrow & K
\end{array}
$$

commutes up to homotopy: $\mu \circ T \simeq \mu$. An $H$-group is an $H$-space $(K, k_0)$ with homotopy associative multiplication $\mu$ and homotopy inverse $\nu$.

All maps and homotopies should be relative to the base point. $K \times K$ has base point $(k_0, k_0)$.

**Definition 2.8** If $X$ and $Y$ are pointed topological spaces, their union (identifying their two base points) in the category of pointed topological spaces will be denoted by $X \vee Y$. If $X$ has base point
$x_0$ and $Y$ has base point $y_0$, $X \vee Y$ may be regarded as the subspace $X \times y_0 \cup x_0 \times Y$ of $X \times Y$. $X \vee Y$ is called the **wedge** (product) of $X$ and $Y$.

**Definition 2.9** An **$H$-cospace** is a pointed space $(K, k_0)$ together with a continuous comultiplication $\mu' : K \rightarrow K \vee K$ such that for the constant map $k_0$, the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{(k_0, 1)} & K \vee K & \xrightarrow{(1, k_0)} & K \\
\downarrow & & \downarrow \mu' & & \downarrow \\
1 & & K & & 1
\end{array}
$$

commutes up to homotopy (here $(k_0, 1)$ denotes the map such that $(k_0, 1)(k, k_0) = k_0$ and $(k_0, 1)(k, k) = k$ for all $k \in K$). $\mu'$ is further required to be homotopy associative—i.e. the diagram

$$
\begin{array}{ccc}
K \vee K \vee K & \xrightarrow{\mu \times 1} & K \vee K \\
\downarrow 1 \vee \mu' & & \downarrow \mu' \\
K \vee K & \xrightarrow{\mu'} & K
\end{array}
$$

must commute up to homotopy. Moreover, $K$ is called an **$H$-cogroup** if $K$ has a continuous homotopy inverse $\nu' : K \rightarrow K$ such that the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\mu} & K \\
\downarrow \nu' & & \downarrow \\
K & \xrightarrow{\mu'} & K
\end{array}
$$
commutes up to homotopy. Finally $K$ is called a homotopy commutative $H$-cogroup if in addition the diagram

$$
\begin{array}{c}
K \xrightarrow{(\nu, 1)} K \cup K \xrightarrow{1, \nu'} K \\
\downarrow \quad \downarrow \\
k_0 & \mu \quad k_0
\end{array}
$$

commutes up to homotopy.

**Proposition 2.10** If $(K, k_0)$ is an $H$-group with multiplication $\mu$ and homotopy inverse $\nu$, then for every $(X, x_0) \in \text{Top}$, the set

$$[X, x_0; K, k_0]$$

can be given the structure of a group.

**Proof.** If we define the product $[f] \cdot [g]$ to be the homotopy class of the composition

$$
\begin{array}{ccc}
X & \stackrel{\triangle}{\longrightarrow} & X \times X \\
& \searrow & \downarrow f \times g \\
& & K \times K \stackrel{\mu}{\longrightarrow} K
\end{array}
$$

Here $\triangle$ is the diagonal map given by $\triangle(x) = (x, x)$. The identity of the group is the class $[k_0]$ of the constant map, and the
inverse is given by \([f]^{-1} = [\nu \circ f]\). If \(\mu\) is homotopy commutative, then \([X, x_0; K, k_0]\) is abelian. Every map \(f : (X, x_0) \to (Y, y_0)\) induces a homomorphism \(f^* : [Y, y_0; K, k_0] \to [X, x_0; K, k_0]\) as follows: For \(\phi \in [Y, y_0; K, k_0]\), \(f^*(\phi)\) is defined to be the composite \(\phi \circ f\).

**Proposition 2.11** If \((K, k_0)\) is an \(H\)-cogroup with comultiplication \(\mu'\) and homotopy inverse \(\nu'\), then for every pointed space \((X, x_0) \in PG\) the set \([K, k_0; X, x_0]\) can be given the structure of group.

**Proof.** If we define the product \([f] \cdot [g]\) to be the homotopy class of the composition

\[
\begin{array}{cccccc}
K & \xrightarrow{\mu'} & K \vee K & \xrightarrow{f \vee g} & X \vee X & \xrightarrow{\triangle'} & X.
\end{array}
\]

Here \(\triangle'\) is the folding map given by \(\triangle'(x, x_0) = x = \triangle'(x_0, x)\). The identity of the group is the class \([x_0]\) of the constant map, and the inverse is given by \([f]^{-1} = [f \circ \nu']\). If \(\mu'\) is homotopy commutative, then \([K, k_0; X, x_0]\) is abelian. Every map \(f : (X, x_0) \to (Y, y_0)\) induces a homomorphism.
Chapter 3. The loop spaces and the suspensions

Definition 3.1 If \((X, x_0)\) is a pointed space, then its **loop space**, denoted by \(\Omega(X, x_0)\) is the function space
\[
\Omega(X, x_0) = \{w : I \to X \mid w(0) = w(1) = x_0\},
\]
topologized as a subspace of \(X^I\) (equipped with the compact-open topology). One usually chooses \(w_0\), the constant path at \(x_0\), as a base point of \(\Omega(X, x_0)\).

If the loop space does not depend on the choice of base point, we often write \(\Omega X\) instead of \(\Omega(X, x_0)\).

Definition 3.2 Let \(X\) and \(Y\) be spaces, and let \(y_0 \in Y\). The constant map at \(y_0\) is the function \(c : X \to Y\) with \(c(x) = y_0\) for all \(x \in X\). A continuous map \(f : X \to Y\) is **nullhomotopic** if there is a constant map \(c : X \to Y\) with \(f \simeq c\).

Let \(f : X \to Y\) be a pointed map, define \(\Omega f : \Omega X \to \Omega Y\) by \(\Omega f(w) = f \circ w\), where \(w\) is a loop in \(X\) (at the base point). Then the loop space defines a functor \(\Omega : \text{Top}_* \to \text{Top}_*\).
Proposition 3.3 If \((X, x_0)\) is a pointed space, then \(\Omega X\) is an \(H\)-group.

Proof. Define the multiplication \(\mu : \Omega X \times \Omega X \to \Omega X\) by

\[(w, w') \mapsto w \ast w\]

where, as usual,

\[
(w \ast w')(t) = \begin{cases} w(2t) & \text{if } 0 \leq t \leq 1/2 \\ w'(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}
\]

To prove the homotopy associativity we may easily construct a homotopy \(H : \Omega X \times \Omega X \times I \to \Omega X\) between two maps; \(\mu (\mu \times 1) = \mu (1 \times \mu )\).

The homotopy inverse \(\nu : \Omega X \to \Omega X\) is defined by \(\nu (w) = w^{-1}\) for \(w \in \Omega X\), where \(w^{-1}(t) = w(1 - t)\).

Definition 3.4 If \((Z, z_0)\) is a pointed space, then the suspension of \(Z\), denoted by \(SZ\), is the quotient space

\[
SZ = (Z \times I) / (Z \times \{0\}) \cup (\{z_0\} \times I),
\]

where the identified subset is regarded as the base point of \(SZ\).

Let \(f : X \to Y\) be a pointed map, define \(Sf : SX \to SY\) by the formula \(Sf([x, t]) = [f(x), t]\). Then the suspension defines a
functor $S : \text{Top}_* \rightarrow \text{Top}_*$.

**Proposition 3.5** For any pointed topological space $(X, x_0)$, the suspension $SX$ is an $H$-cogroup.

**Proof.** The comultiplication $\mu' : SX \rightarrow SX \vee SX$ is defined by

$$
\mu'([x, t]) = \begin{cases} 
([x, 2t], x_0) & 0 \leq t \leq 1/2 \\
(x_0, [x, 2t - 1]) & 1/2 \leq t \leq 1 
\end{cases}.
$$

The homotopy associativity and the homotopy inverse are trivially shown.

**Lemma 3.6** For all $n \geq 0$, the suspension of sphere $SS^n$ is homeomorphic to $S^{n+1}$.

**Proof.** Let $\rho_0 = (1, 0, \ldots, 0)$ be the base point of $S^n$. We regard $\mathbb{R}^{n+1}$ as imbedded in $\mathbb{R}^{n+2}$ as the set of points in $\mathbb{R}^{n+2}$ whose $(n+2)$nd coordinate is 0. Then $S^n$ is imbedded as an equator in $S^{n+1}$

$$
S^n = \{ z \in \mathbb{R}^{n+2} | \ | z \ | = 1 \text{ and } z_{n+2} = 0 \}
$$

and $\mathbb{E}^{n+1}$ is also imbedded in $\mathbb{E}^{n+2}$:

$$
\mathbb{E}^{n+1} = \{ z \in \mathbb{R}^{n+2} | \ | z \ | \leq 1 \text{ and } z_{n+2} = 0 \}
$$

Let $H_+$ and $H_-$ be the two closed hemispheres of $S^{n+1}$ defined
by the equator $S^n$. Then
\[ H_+ = \{ z \in S^{n+1} | z_{n+2} \geq 0 \} \quad \text{and} \quad H_- = \{ z \in S^{n+1} | z_{n+2} \leq 0 \} \]

and \[ S^{n+1} = H_+ \cup H_- \quad \text{and} \quad S^n = H_+ \cap H_- . \] Furthermore, the projection map \( \mathbb{R}^{n+2} \to \mathbb{R}^{n+1} \) defines projection maps \( p_+: H_+ \to \mathbb{E}^{n+1} \) and \( p_-: H_- \to \mathbb{E}^{n+1} \), which are homeomorphisms.

A map \( f: S(S^n) \to S^{n+1} \) is defined by
\[
  f([z,t]) = \begin{cases} 
    p_+^{-1}(2tz + (1 - 2t)p_0) & 0 \leq t \leq 1/2 \\
    p_+^{-1}((2 - 2t)z + (2t - 1)p_0) & 1/2 \leq t \leq 1 
  \end{cases}
\]

and is verified to be a homeomorphism \( f: S(S^n) \approx S^{n+1} \).

**Definition 3.7** For every pointed space \((X, x_0)\) and every \( n \geq 0 \),
\[
\pi_n(X, x_0) := [(S^n, s_n), (X, x_0)].
\]

We shall usually abbreviate \( \pi_n(X, x_0) \) to \( \pi_nX \) which is called the (higher) homotopy group of \( X \).

Since the sphere \( S^n \) is the suspension of the sphere \( S^{n-1} \), \( S^n \) is an \( H \)-cogroup so that \( \pi_n(X, x_0) := [(S^n, s_n), (X, x_0)] \) forms a group. The following Lemma is an immediate consequence of Lemma 3.6.

**Lemma 3.8** For every pointed space \( X \), \( \pi_n(X) \) is a group for all \( n \geq 1 \).
Theorem 3.9 If $Q$ is an $H$-cogroup and $P$ is an $H$-group, then group operations on $[Q,P]$ determined by the comultiplication $m$ of $Q$ and by the multiplication $\mu$ of $P$ coincide.

Proof. Let $f,g : Q \to P$. The following diagram commutes up to homotopy;

$$
\begin{array}{c}
Q \xrightarrow{m} Q \vee Q \xrightarrow{f \vee g} P \vee P \\
\downarrow k \downarrow k \downarrow \triangledown_p \\
Q \times Q \xrightarrow{f \times g} P \times P \xrightarrow{\mu} P
\end{array}
$$

But the multiplication determined by $m$ is $[f] \ast [g] = [(f,g)m] = [\triangledown (f \vee g)m]$, and that determined by $\mu$ is $[f] \circ [g] = [\mu (f,g)] = [\mu (f \times g) \triangle]$. Hence $[f] \ast [g] = [f] \circ [g]$.  

\[\Box\]

Theorem 3.10 For all pointed topological spaces $X$ and $Y$, there is a natural one-to-one correspondence

$$\Phi : [SX,Y] \to [X,\Omega Y].$$

Proof. Define $\Phi : [SX,Y] \to [X,\Omega Y]$ as follows: For $\phi \in [SX,Y]$ and $x \in X$, $\Phi(\phi)(x) \in \Omega Y$ is defined by $\Phi(\phi)(x)(t) = \phi([x,t])$ where $[x,t] \in SX$. Here all maps stand for the homotopy classes to which they belong.
We define the inverse map of $\Phi$. Define $\Psi : [X, \Omega Y] \to [SX, Y]$ as follows: For $\psi \in [X, \Omega Y]$ and $x \in X$, $\Psi(\psi)([x, t]) \in Y$ is defined to be $\psi(x)(t)$, where $[x, t] \in SX$.

The map $\Phi : [SX, Y] \to [X, \Omega Y]$ is actually a group isomorphism.

**Proposition 3.11** The correspondence $\Phi : [SX, Y] \to [X, \Omega Y]$ is an isomorphism of groups.

**Proof.** Given $f, g : (SX, \ast) \to (Y, y_0)$ let $f \circ g$ denote the composite

$$SX \xrightarrow{\mu} SX \vee SX \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\triangle'} Y.$$ 

Given $f', g' : (X, x_0) \to (\Omega Y, w_0)$ let $f' \ast g'$ denote the composite

$$X \xrightarrow{\triangle} X \times X \xrightarrow{f' \times g'} \Omega Y \to \Omega Y \xrightarrow{\mu} \Omega Y.$$ 

We shall prove that $(f \circ g) = \hat{f} \ast \hat{g}$. For all $x \in X$, $t \in I$ we have

$$(f \circ g)(x)(t) = (f \circ g)[t, x] = \begin{cases} f([2t, x]) & 0 \leq t \leq 1/2 \\ g([2t - 1, x]) & 1/2 \leq t \leq 1 \end{cases}.$$ 

$$(\hat{f} \circ \hat{g})(x)(t) = \begin{cases} (\hat{f}(x))(2t) & 0 \leq t \leq 1/2 \\ (\hat{g}(x))(2t - 1) & 1/2 \leq t \leq 1 \end{cases} = \begin{cases} f([2t, x]) & 0 \leq t \leq 1/2 \\ g([2t - 1, x]) & 1/2 \leq t \leq 1 \end{cases}.$$
We now have the following very useful result.

**Corollary 3.12** If $X$ is a pointed space, then

$$\pi_n(X) \cong \pi_{n-k}(\Omega^k X)$$

for all $1 \leq k \leq n-1$ (where $\Omega^k$ is the composite of $\Omega$ with itself $k$ times). In particular, if $n \geq 2$,

$$\pi_n(X) \cong \pi_1(\Omega^{n-1} X).$$

**Proof.**

$$\pi_n(X) = [S^n, X] = [S^n S^0, X]$$

$$= [S^{n-k} S^0, \Omega^k X] = [S^{n-k}, \Omega^k X] = \pi_{n-k}(\Omega^k X). \quad \blacksquare$$
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