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敎育學碩士學位請求論文

$T_n$ - 空間에 대한 研究

On $T_n$ - spaces

仁荷大學校 教育大學院

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1
教育학碩士學位請求論文

$T_n$ - 空間에 대한 研究

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이 論文을 碩士學位 論文으로 提出함.
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2008년 2월

主審

副審

副審
국문초록

본 논문에서는 분리공리에 따른 여러 가지 위상공간의 분류에 대해 연구하였다.

$T_i$-공리에 만족하는 위상공간은 $T_i$-공간이라 부른다 ($i=0,1,2,3,4$)。
 이러한 공간들의 위상적 성질을 연구하고 다양한 예를 알아보았다.

$T_0$, $T_1$와 $T_2$-공간 및 $T_{1\frac{1}{3}}$-공간과 $T_{2\frac{1}{3}}$-공간의 정의를 소개하고 $T_{2\frac{1}{3}}$-공간이 $T_{1\frac{1}{3}}$-공간임을 증명하였다.
정규공간($T_3$-공간), Urysohn 공간($T_{1\frac{1}{2}}$-공간), Tychonoff 공간($T_{1\frac{3}{2}}$-공간), 정칙공간($T_{4\frac{1}{2}}$-공간)과 완전정규공간($T_{4\frac{3}{2}}$-공간) 등도 소개하고 주요 성질들을 알아보았다.
Abstract

We have studied the classification of various topological spaces depending on separation axioms.

The spaces satisfying $T_i$ axioms are called $T_i$-spaces for $i = 0, 1, 2, 3, 4$. We studied the topological properties of those spaces and give various examples of such spaces.

We introduced and studied main properties of $T_0$, $T_1$, $T_{\frac{1}{3}}$, $T_{\frac{2}{3}}$, $T_{\frac{3}{2}}$, $T_{\frac{4}{3}}$, $T_{\frac{5}{2}}$, $T_{\frac{6}{3}}$, $T_{\frac{7}{2}}$, $T_{\frac{8}{3}}$, $T_{\frac{9}{2}}$, $T_{\frac{10}{3}}$, $T_{\frac{11}{2}}$, $T_{\frac{12}{3}}$, $T_{\frac{13}{2}}$, $T_{\frac{14}{3}}$, $T_{\frac{15}{2}}$.
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0. INTRODUCTION

The euclidean space $\mathbb{R}^n$ is the most classical and important topological space. It is, in the topological point of view, a very good space, which means that it satisfies many important topological properties, for example, separation axioms. The euclidean space is not only a Hausdorff space, but also a normal space. A space $X$ is said to be Hausdorff if for each pair $x, y$ of distinct points of $X$, there exist disjoint open sets containing $x$ and $y$, respectively, that is, a Hausdorff space satisfies a separation axiom for a pair of distinct two points.

The axioms are called separation axioms for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets. A space is said to be regular if it satisfies the separation axiom between a point and a disjoint closed set. A space is said to be normal if it satisfies the separation axiom between two disjoint closed sets. It is clear that a Hausdorff space is regular as long as one point sets are closed. Note, however, that two-point space in the trivial topology satisfies both the regularity and normality axioms, but it is not Hausdorff.

The Hausdorff, regularity and normality axioms are called $T_2$, $T_3$ and $T_4$ axioms. The spaces satisfying $T_i$ axioms are called $T_i$-spaces for $i = 0, 1, 2, 3, 4$. A space $X$ is a $T_1$-space if every one point subset is closed.

The $T_0$ property was introduced by A. N. Kolmogorov, and the $T_1$ property was introduced by Frechet in 1907. Frechet used the name
accessible spaces for $T_1$-spaces. Hausdorff introduced the $T_2$ property in 1914. $T_0$ and $T_1$ axioms are weaker than the $T_2$ property.

In chapter 1, we introduce $T_0$, $T_1$ and $T_2$-spaces. We also introduce some spaces which are between $T_1$ and $T_2$. $T_2$-spaces are most important out of these spaces. We give, in this chapter, some important theorems with respect to $T_2$-spaces. We also introduce the definitions of $T_{\frac{1}{3}}$-spaces and $T_{\frac{2}{3}}$-spaces. We provide a new own proof that each $T_{\frac{2}{3}}$-space is a $T_{\frac{1}{3}}$-space (Theorem 1.18).

In chapter 2, we will deal with regular spaces which are $T_3$-spaces. Regular spaces were first studied by Vietoris in 1921. We usually require that a regular space be a $T_1$ so that every regular space is a Hausdorff space. We introduce Urysohn spaces which are weaker than regular spaces, but stronger than Hausdorff spaces. A Urysohn space is also called a $T_{\frac{1}{2}}$-space. A space $X$ is completely regular or a Tychonoff space if one-point sets are closed in $X$ and if for each point $x_0$ and each closed set $A$ not containing $x_0$, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$. A completely regular space is also called a $T_{\frac{3}{2}}$-space. We show that a completely regular space is regular (Theorem 2.13).

In chapter 3, we will deal with normal spaces which are $T_4$-spaces. Normal spaces were introduced by Vietoris in 1921 and by Tietze in 1923. They were also studied independently by Alexandroff and
Urysohn in 1925. We require that a normal space be a $T_1$-space so that every normal space is a regular space. We show that a metrizable space is normal (Theorem 3.7) and also show that every compact Hausdorff space is normal (Theorem 3.8). Finally, we introduce a space even stronger than a normal space, called completely normal space (introduced by Tietz). We show that a space $X$ is completely normal if and only if every subspace of $X$ is normal (Theorem 3.12).
Chapter 1. $T_0$, $T_1$, and $T_2$-SPACES

In this chapter, we introduce three separation axioms and explore their properties. We provide general examples for comprehension of definitions and properties.

**Definition 1.1** A space $X$ is a $T_0$-space if for each pair $x, y$ of distinct points of $X$, there exists an open set containing one of the points but not the other.

**Definition 1.2** A space $X$ is a $T_1$-space if for each pair $x, y$ of distinct points of $X$, there exist two open sets $U$ and $V$ in $X$ such that $x$ belongs to $U$ but $y$ does not, and $y$ belongs to $V$ but $x$ does not.

**Theorem 1.3** A space $X$ is a $T_1$-space if and only if each finite subset of $X$ is closed.

**Proof.** Suppose $X$ is $T_1$. It is sufficient to prove that each one point set $\{x\}$ is closed since any finite set is the union of a finite number of such sets. If $y$ is a point of $X$ different from $x$, there is an open set $V$ containing $y$ but not $x$. There is no limit point of $\{x\}$, so $\{x\}$ is a closed set.

Conversely, suppose each finite subset of $X$ is closed and consider distinct points $x, y$ in $X$. Then

$$U = X - \{y\}, \quad V = X - \{x\}$$
are open sets, $U$ contains $x$ but not $y$, and $V$ contains $y$ but not $x$. Thus $X$ is $T_1$.

**Theorem 1.4** Let $X$ be a space satisfying $T_1$ axiom. Let $A$ be a subset of $X$. Then the point $x$ is a limit point of $A$ if and only if every neighborhood of $x$ contains infinitely many points of $A$.

**Proof.** If every neighborhood of $x$ intersects $A$ in infinitely many points, it certainly intersects $A$ in some point other than $x$ itself, so that $x$ is a limit point of $A$.

Conversely, suppose that $x$ is a limit point of $A$, and suppose some neighborhood $U$ of $x$ intersects $A$ in only finitely many points. Then $U$ also intersects $A-\{x\}$ at finitely many points; let $x_1,\ldots,x_m$ be points of $U \cap (A-\{x\})$. The set $X-\{x_1,\ldots,x_m\}$ is an open set of $X$, since the finite point set $\{x_1,\ldots,x_m\}$ is closed.

Then $U \cap (X-\{x_1,\ldots,x_m\})$ is a neighborhood of $x$ that intersects the set $A-\{x\}$ not at all. This contradicts the assumption that $x$ is a limit point of $A$.

**Definition 1.5** A space $X$ is a **$T_2$-space** or **Hausdorff space** if for each pair $x$, $y$ of distinct points of $X$, there are disjoint open sets $U$ and $V$ such that $x$ belongs to $U$ and $y$ belongs to $V$.

$T_2$ axiom implies $T_1$ axiom, hence in a Hausdorff space every finite set is closed.
Theorem 1.6  A subspace of a Hausdorff space is a Hausdorff space.

Proof. Let $X$ be a Hausdorff space. Let $x$ and $y$ be two points of the subspace $Y$ of $X$. If $U$ and $V$ are disjoint neighborhoods in $X$ of $x$ and $y$, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhood of $x$ and $y$ in $Y$. ■

Theorem 1.7  If $X$ is a Hausdorff space, then a sequence of points of $X$ converges to at most one point of $X$.

Proof. Suppose that $x_n$ is a sequence of points of $X$ that converges to $x$. If $y \neq x$, there exist $U$ and $V$ be disjoint neighborhood of $x$ and $y$, respectively. Since $U$ contains $x_n$ for all but finitely many values of $n$, the set $V$ cannot. Therefore, $x_n$ cannot converge to $y$. ■

Theorem 1.8  Every compact subspace of a Hausdorff spaces is closed.

Proof. Let $Y$ be a compact subspace of the Hausdorff space $X$. We show that $Y$ is closed. i.e. $X - Y$ is open.

Let $x_0$ be a point of $X - Y$. We show that there is a neighborhood of $x_0$ that is disjoint from $Y$. For each point $y$ of $Y$, let us choose disjoint neighborhoods $U_y$ and $V_y$ of the points $x_0$ and $y$, respectively. The collection $\{V_y \mid k \in Y\}$ is covering of $Y$ by sets open in $X$; therefore, finitely many of them $V_{y_1}, \ldots, V_{y_n}$ cover $Y$. The open set
\[ V = V_{y_1} \cup \cdots \cup V_{y_n} \]
contains \( Y \), and it is disjoint from the open set
\[ U = U_{y_1} \cap \cdots \cap U_{y_n} \]
formed by taking the intersection of the corresponding neighborhoods of \( x_0 \). For if \( z \) is a point of \( V \), then \( z \in V_{y_i} \) for some \( i \), hence \( z \notin U_{y_i} \) and so \( z \notin U \). Then \( U \) is a neighborhood of \( x_0 \) disjoint from \( Y \), as desired.

\[ \blacksquare \]

**Theorem 1.9** The product of Hausdorff spaces is a Hausdorff.

**Proof.** Let \( \{X_\alpha | \alpha \in \mathbb{A}\} \) be a family of Hausdorff spaces. Let \( x = (x_\alpha) \) and \( y = (y_\alpha) \) be distinct points of the product space \( X = \prod_{\alpha \in \mathbb{A}} X_\alpha \). Then there is some index \( \beta \in \mathbb{A} \) such that \( x_\beta \neq y_\beta \). Since \( X_\beta \) is Hausdorff, there exist disjoint open sets \( U_\beta \) and \( V_\beta \) in \( X_\beta \) containing \( x_\beta \) and \( y_\beta \), respectively. Then
\[ U = \pi_{\beta}^{-1}(U_\beta), \quad V = \pi_{\beta}^{-1}(V_\beta) \]
are disjoint open sets in \( X \) containing \( x \) and \( y \), respectively.

\[ \blacksquare \]

\( T_0, T_1, T_2 \) is the arrangement of the properties in order of increasing strength. The following examples show that a \( T_0 \)-space may fail to be \( T_1 \) and that \( T_1 \)-space may fail to be \( T_2 \).

**Example 1.10** A \( T_0 \)-space which is not \( T_1 \).

Let \( X = \{a, b\} \) be a two-point set with open sets \( \emptyset, \{a\}, \) and \( X \).
Then given two distinct points of $X$, one of them (namely $a$) is contained in an open set which does not contain the other. However, every open set containing $b$ also contains $a$, so $X$ is not $T_1$.

**Example 1.11** A $T_1$-space which is not $T_2$.

Let $X$ denote the set of real numbers with the finite complement topology. Then for distinct points $a, b$ in $X$,

$$U = X - \{b\}, \quad V = X - \{a\}$$

are open sets such that $U$ containing $a$ but not $b$ and $V$ contains $b$ but not $a$. Thus $X$ is $T_1$. As a matter of fact, the finite complement topology is the smallest topology satisfying $T_1$ axiom. Meanwhile, $X$ is not $T_2$, because any two nonempty open subsets are not disjoint.

We give an example of $T_2$-space.

**Example 1.12** Every metric space is Hausdorff.

If $x$ and $y$ are distinct points then $\rho(x, y) = \epsilon > 0$, so the disk $U(x, \frac{\epsilon}{2})$ and $U(y, \frac{\epsilon}{2})$ are disjoint open sets containing $x$ and $y$ respectively.

**Theorem 1.13** Let $X$ be a space, $Y$ be a $T_2$ space and $f, g : X \to Y$ be continuous functions, then

(a) $\{x \in X | f(x) = g(x)\}$ is a closed subset of $X$.

(b) If $f$ and $g$ agree on a dense subset of $X$, then $f = g$. 
Proof. (a) Let $A = \{ x \in X \mid f(x) = g(x) \}$. Then it is sufficient to show that $X - A$ is an open set. For each point $x_0$ in $X - A$, $f(x_0) \neq g(x_0)$. Since $Y$ is a $T_2$-space, there exist disjoint open sets $U, V$ such that $f(x_0) \in U$, $g(x_0) \in V$ and because $f$, $g$ are continuous functions, $f^{-1}(U) \cap g^{-1}(V)$ is an open set containing $x_0$.

Now, we show that $f^{-1}(U) \cap g^{-1}(V) \subset (X - A)$. Let a point $x \in f^{-1}(U) \cap g^{-1}(V)$. Then $f(x) \in U$, $g(x) \in V$ and $U \cap V = \emptyset$. Hence $f(x) \neq g(x)$ so $X \in X - A$. Therefore $X - A$ is open. 

(b) Let $A = \{ x \in X \mid f(x) = g(x) \}$ and $D$ be a dense subset of $X$ such that $f|_D = g|_D$. Then $D \subset A$ and $A$ is a closed set. Since $\overline{D}$ is the smallest closed subset of $X$ containing $D$, and $D$ is a dense subset of $X$, we have $\overline{D} = X \subset A$. Thus $A = X$. For each point $x$ in $X$, $f = g$. ■

Theorem 1.14 Let $f : X \to Y$ be a continuous function Assume that $Y$ is $T_2$, then $\{ (x_1, x_2) \mid f(x_1) = f(x_2) \}$ is a closed subset of $X \times X$. The converse holds if $f$ is an open map.

Proof. Let $A = \{ (x_1, x_2) \mid f(x_1) = f(x_2) \}$, and suppose $(x_1, x_2) \notin A$. Then $f(x_1) \neq f(x_2)$. Since $Y$ is Hausdorff, there exist disjoint open sets $U$ and $V$ such that $f(x_1) \in U$ and $f(x_2) \in V$. Since $f$ is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open. Hence $f^{-1}(U) \times f^{-1}(V)$ is a neighborhood of $(x_1, x_2)$. Notice that if $(x_1, x_2) \notin f^{-1}(U) \times f^{-1}(V)$, then $f(x_1) \in U$ and $f(x_2) \in V$ so $f(x_1) \neq f(x_2)$. Thus $f^{-1}(U) \times f^{-1}(V) \cap A = \emptyset$, and $(X \times X) - A$ is open.
Conversely, let $y_1 \neq y_2 \in Y$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$, then $(x_1, x_2) \in X - A$. Since $A$ is closed, there is a basis element $U \times V$ such that $(x_1, x_2) \in U \times V \subset X - A$. Since $f$ is open, $f(U)$ and $f(V)$ are disjoint open sets containing $y_1$ and $y_2$, respectively. ■

**Corollary 1.15** A space $X$ is a Hausdorff space if and only if

$$\Delta = \{ (x, x) \mid x \in X \} = \{ (x_1, x_2) \mid x_1 = x_2 \}$$

is a closed subset of $X \times X$.

**Definition 1.16** A $T_{1\frac{1}{3}}$-space is a space in which each sequence has at most one limit.

Theorem 1.8 implies that each $T_1$-space is $T_{1\frac{1}{3}}$. It is easy to see that a $T_{1\frac{1}{3}}$-space is $T_1$; suppose it is not $T_1$, that is, there is a pair of distinct points $x_1$, $x_2$ such that every open set containing one contains another. Then the sequence $x_1, x_1, x_2, \ldots$ converges both to $x_1$ and $x_2$.

**Definition 1.17** A $T_{2\frac{1}{3}}$-space is a space in which each compact set is closed.

Theorem 1.9 implies that a $T_2$-space is $T_{2\frac{1}{3}}$.

**Theorem 1.18** A $T_{2\frac{1}{3}}$-space is $T_{1\frac{1}{3}}$-space.
Proof. Let $X$ be a $T_{\frac{2}{3}}$-space. Suppose $X$ is not a $T_{\frac{1}{3}}$-space. Let $\{x_n\}$ be a sequence in $X$ converging to $n$ distinct points $z_1, z_2, \ldots, z_n (n \geq 2)$. Then the set $A = \{x_n \mid n \in \mathbb{Z}_+\} \cup \{z_1\} - \{z_2, z_3, \ldots, z_n\}$ is compact, because any open set containing $z_1$ contains all points of $A$ except finite points. Note, however, that $A$ is not closed, so $X$ is not a $T_{\frac{1}{3}}$-space. 

\[\square\]
Chapter 2. REGULAR SPACES

The properties $T_0$, $T_1$, $T_2$ describe the separation of pairs of points by open sets. The properties that are studied in this chapter describe the separation of a point from a closed set by open sets and, hence, are more restrictive.

**Definition 2.1** Suppose that one-point sets are closed in $X$. Then $X$ is said to be $T_3$-space or regular space if for each pair consisting of a point $x$ and a closed set $C$ disjoint from $x$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $C \subset V$.

If $X$ is $T_3$ and $x$, $y$ are distinct points of $X$, then $C = \{y\}$ is a closed set which does not contain $x$. Thus there are disjoint open sets $U$ and $V$ with $x \in U$ and $y \in V$. Thus $T_3$-space is $T_2$.

The following example shows that the regularity axiom is stronger than the Hausdorff axiom.

**Example 2.2** The space $\mathbb{R}_K$ is Hausdorff but not regular.

Recall that $\mathbb{R}_K$ denotes the reals in the topology having as basis all open intervals $(a,b)$ and all sets of the form $(a, b) - K$, where $K = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}$. This space is Hausdorff, because any two distinct points have disjoint open intervals containing them.

But it is not regular. The set $K$ is closed in $\mathbb{R}_K$, and it does not
contain the point 0. Suppose that there exist disjoint open sets $U$ and $V$ containing 0 and $K$, respectively. Choose a basis element containing 0 and lying in $U$. It must be a basis element of the form $(a, b) - K$, since each basis element of the form $(a, b)$ containing 0 intersects $K$. Choose $n$ large enough that $\frac{1}{n} \in (a, b)$. Then choose a basis element about $\frac{1}{n}$ contained in $V$; it must be a basis element of the form $(c, d)$. Finally, choose $z$ so that $z < \frac{1}{n}$ and $z > \max\left\{e, \frac{1}{(n+1)}\right\}$. Then $z$ belongs to both $U$ and $V$, so they are not disjoint.

**Theorem 2.3** Let $X$ be a topological space. Suppose every one-point set in $X$ be closed. $X$ is regular if and only if given a point $x$ of $X$ and a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $V \subset U$.

**Proof.** Suppose that $X$ is regular, and suppose that the point $x$ and the neighborhood $U$ of $x$ are given. Then $X - U$ is a closed set which does not contain $x$. By hypothesis, there exist disjoint open sets $V$ and $W$ such that

$$x \in V, \ (X - U) \subset W.$$

Since $V \subset (X - W)$ and $X - W$ is closed, then $\overline{V} \subset (X - W)$. Thus

$$\overline{V} \subset (X - W) \subset X - (X - U) = U,$$

so $V$ is the required open set.

To prove the converse, suppose a point $x$ and a closed set $C$ not
containing $x$ are given. Let $U = X - C$, then $U$ is an open set containing $x$. By hypothesis, there is a neighborhood $V$ of $x$ such that $\overline{V} \subset U$. The open sets $V$ and $X - \overline{V}$ are disjoint open sets containing $x$ and $C$, respectively. Thus $X$ is regular.

**Theorem 2.4** Let $X$ be a topological space. Suppose every one-point set in $X$ be closed. $X$ is regular if and only if given a point $x$ of $X$ and closed set $C$ not containing $x$, there exist open sets $U$ and $V$ such that $x \in U$, $C \subset V$, and $\overline{U}$ and $\overline{V}$ are disjoint.

**Proof.** Suppose that $x$ is a point and $C$ is a closed set which does not contain $x$. By Theorem 2.1 there is an open set $W$ such that $x \in W$, $\overline{W} \subset (X - C)$. Applying the same theorem again, there is an open set $U$ containing $x$ with $\overline{U} \subset W$. Let $V = X - \overline{W}$. Then

\[ \overline{U} \subset W \subset \overline{W} \subset (X - C), \]

so

\[ C \subset (X - \overline{W}) = V. \]

Since

\[ \overline{U} \cap \overline{V} = \overline{U} \cap (X - \overline{W}) \subset W \cap (X - \overline{W}) = \emptyset, \]

then $U$ and $V$ are the required open sets. 

**Theorem 2.5** A subspace of regular space is regular.

**Proof.** Let $Y$ be a subspace of the regular space $X$. Then one-point sets are closed in $Y$. Let $x$ be a point of $Y$ and let $C$ be a closed
subset of $Y$ disjoint from $x$. Now $\overline{C} \cap Y = C$, where $\overline{C}$ denotes the closure of $C$ in $X$. Therefore, $x \not\in \overline{C}$, so, using regularity of $X$, we can choose disjoint open sets $U$ and $V$ of $X$ containing $x$ and $\overline{C}$, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in $Y$ containing $x$ and $C$, respectively. ■

Theorem 2.6 A product of regular space is regular.

The proof of Theorem 2.6 is based upon the following result.

Lemma 2.7 Let $\{X_{\alpha}\}$ be an indexed family of spaces; let $A_{\alpha} \subset X_{\alpha}$ for each $\alpha$. If $\prod X_{\alpha}$ is given either the product or the box topology, then $\prod A_{\alpha} = \prod A_{\alpha}$.

Proof. Let $x = (x_{\alpha})$ be a point of $\prod A_{\alpha}$; we show that $x \in \prod A_{\alpha}$. Let $U = \prod U_{\alpha}$ be a basis element for either the box or product topology that contains $x$. Since $x_{\alpha} \in A_{\alpha}$, we can choose a point $y_{\alpha} \in U_{\alpha} \cap A_{\alpha}$ for each $\alpha$. Then $y = (y_{\alpha})$ belongs to both $U$ and $\prod A_{\alpha}$. Since $U$ is arbitrary, it follows that $x$ belongs to the closure of $\prod A_{\alpha}$.

Conversely, suppose $x = (x_{\alpha})$ lies in the closure of $\prod A_{\alpha}$, in either topology. We show that for any given index $\beta$, we have $x_{\beta} \in \overline{A_{\beta}}$. Let $V_{\beta}$ be an arbitrary open set of $X_{\beta}$ containing $x_{\beta}$. Since $\pi^{-1}_{\beta}(V_{\beta})$ is open in $\prod X_{\alpha}$ in either topology, it contains a point $y = (y_{\alpha})$ of $\prod A_{\alpha}$. Then
$y_\beta$ belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \subseteq \overline{A_\beta}.$

**Proof of Theorem 2.6** Let $\{X_\alpha\}$ be a family of regular spaces; let $X = \prod_{\alpha \in A} X_\alpha$. By Theorem 1.9, $X$ is Hausdorff, so that one-point sets are closed in $X$. Let $x = (x_\alpha)$ be a point of $X$ and let $U$ be a neighborhood of $x$ in $X$. Choose the basis element $\prod U_\alpha$ about $x$ contained in $U$. Choose, for each $\alpha$, a neighborhood $V_\alpha$ of $x_\alpha$ in $X_\alpha$ such that $\overline{V_\alpha} \subseteq U_\alpha$; if it happen that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of $x$ in $X$. Since $V = \prod V_\alpha$ by Lemma 2.1, it follows at once that $\overline{V} \subseteq \prod U_\alpha \subseteq U$, so that $X$ is regular.

**Theorem 2.8** If $X$ is $T_3$ and $f$ is a continuous, open and closed map of $X$ onto $Y$, then $Y$ is $T_2$.

**Proof.** By Corollary 1.15 it is sufficient to show that the set

$$A = \{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\}$$

is closed in $X \times X$. If $(x_1, x_2) \not\in A$, then $x_1 \in f^{-1}[f(x_2)]$ so that, since $X$ is regular, there are disjoint open sets $U$ and $V$ with $x \in U$ and $f^{-1}[f(x_2)] \subseteq V$. Since $f$ is closed, we can find a saturated open set in $X$ containing $f^{-1}[f(x_2)]$ and contained in $V$; that is $f^{-1}[f(x_2)] \subseteq f^{-1}(W) \subseteq V$ for some open set in $W$ in $Y$. Then $U \times f^{-1}(W)$ is a neighborhood of $(x_1, x_2)$ which cannot meet $A$, since $U \cap f^{-1}(W) = \emptyset$. 

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Definition 2.9 A space $X$ is a $T_{\frac{1}{2}}$-space or Urysohn space provided that for each pair $x, y$ of distinct point of $X$, there exist open sets $U$ and $V$ with disjoint closures such that $x\in U$ and $y\in V$.

Theorem 2.4 implies that the following Lemma.

Lemma 2.10 Every regular space is a Urysohn space.

It is obvious that every Urysohn space is Hausdorff.

Definition 2.11 If $A$ and $B$ are two subsets of the topological space $X$, and if there is a continuous function $f : X \to [0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$, we say that $A$ and $B$ can be separated by a continuous function.

Definition 2.12 A space $X$ is completely regular or a Tychonoff space if one-point sets are closed in $X$ and if for each point $x_0$ and each closed set $A$ not containing $x_0$, there is a continuous function $f : X \to [0,1]$ such that $f(x_0)=1$ and $f(A)=\{0\}$.

A completely regular space is also called a $T_{\frac{3}{2}}$-space.

Theorem 2.13 Every completely regular space is regular.

Proof. Let $X$ be a completely regular space, let $C$ be a closed subset of $X$ and let $p \in X - C$. Let $f : X \to [0,1]$ be a continuous function such
that \( f(p) = 1 \) and \( f(x) = 0 \) for all \( x \in C \). Then \( f^{-1}([0, \frac{1}{3}]) \) and \( f^{-1}([\frac{2}{3}, 1]) \) are disjoint open subsets of \( X \) such that \( C \subseteq f^{-1}([0, \frac{1}{3}]) \) and \( p \in f^{-1}([\frac{2}{3}, 1]) \). Therefore \( X \) is regular.

\[ \blacksquare \]

**Theorem 2.14** Every locally compact Hausdorff space is regular.

**Proof.** Suppose \( X \) is locally compact. Let \( x \) be a point of \( X \) and let \( U \) be a neighborhood of \( x \). Take the one-point compactification \( Y \) of \( X \), and let \( C \) be the set \( Y - U \). Then \( C \) is closed in \( Y \), so that \( C \) is a compact subspace of \( Y \). We can choose disjoint open sets \( V \) and \( W \) containing \( x \) and \( C \), respectively. Then the closure \( \overline{V} \) of \( V \) in \( Y \) is compact.

Furthermore, \( V \) is disjoint from \( C \), so that \( \overline{V} \subseteq U \), as desired. And suppose the point \( x \) and the closed set \( B \) not containing \( x \) are given. Let \( U = X - B \). There is a neighborhood \( V \) of \( x \) such that \( \overline{V} \subseteq U \). The open sets \( V \) and \( X - \overline{V} \) are disjoint open sets containing \( x \) and \( B \), respectively. Thus \( X \) is regular.

\[ \blacksquare \]

**Theorem 2.15** A subspace of a completely regular space is completely regular.

**Proof.** Let \( X \) be a completely regular, let \( Y \) be a subspace of \( X \). Let \( x_0 \) be a point of \( Y \), and let \( A \) be a closed set of \( Y \) disjoint from \( x_0 \). Now \( A = \overline{A} \cap Y \), where \( \overline{A} \) denotes the closure of \( A \) in \( X \). Therefore, \( x_0 \in \overline{A} \).
Since $X$ is completely regular, we can choose a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$. The restriction of $f$ to $Y$ is the desired continuous function on $Y$. ■

**Theorem 2.16** A product of completely regular spaces is completely regular.

**Proof.** Let $X = \prod X_\alpha$ be a product of completely regular spaces. Let $b = (b_\alpha)$ be a point of $X$ and let $A$ be a closed set of $X$ disjoint from $b$. Choose a basis element $\prod U_\alpha$ containing $b$ that does not intersect $A$; then $U_\alpha = X_\alpha$ except for finitely many $\alpha$, say $\alpha = \alpha_1, \cdots, \alpha_n$. Given $i = 1, \cdots, n$, choose a continuous function $f_i : X_\alpha \to [0, 1]$ such that $f_i(b_\alpha) = 1$ and $f_i(X - U_\alpha) = \{0\}$. Let $\phi_i(x) = f_i(\pi_\alpha(x))$; then $\phi_i$ maps $X$ continuously into $\mathbb{R}$ and vanishes outside $\pi_\alpha^{-1}(U_\alpha)$.

The product

$$f(x) = \phi_1(x) \cdot \phi_2(x) \cdots \cdot \phi_n(x)$$

is the desired continuous function on $X$, for it equals 1 at $b$ and vanishes outside $\prod U_\alpha$. ■
Chapter 3. NORMAL SPACES

We studied the separation of a closed set and a point in the previous chapter. The two separation properties studied in this chapter are concerned with the separation of two closed sets.

**Definition 3.1** Suppose that one-point sets are closed in \( X \). Then \( X \) is said to be a \( T_4 \)-space or normal space if for each pair \( A, B \) of disjoint open sets containing \( A \) and \( B \), respectively.

We require that a normal space be a \( T_1 \)-space so that every normal space is a regular space. The following is an example of a topological space that is not regular but does have the property that any two disjoint closed sets can be separated by disjoint open sets.

**Example 3.2** If \( X \) is the real line with the topology in which open sets are the sets \( (a, \infty) \) for \( a \in X \), then \( X \) is normal since no two nonempty closed sets are disjoint. But \( X \) is not regular since the point 1 cannot be separated from the closed set \( (-\infty, 0] \) by disjoint open sets.

Since each one-point set in a normal space is closed, it follows easily that every normal space is regular.

**Example 3.3** Consider the topology \( \mathcal{J} = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\} \) on the set
X={a,b,c}, observe that the closed sets are X, ∅, {b,c}, {a,c} and {c}. If $F_1$ and $F_2$ are disjoint closed subsets of $(X, τ)$, then one of them, say $F_1$, must be the empty set ∅. Hence ∅ and X are disjoint open sets and $F_1 ⊂ ∅$, $F_2 ⊂ X$. In the words, $(X, τ)$ is a normal space. On the other hand $(X, τ)$ is not a $T_3$-space since the one-point set {a} is not closed. Furthermore, $(X, τ)$ is not regular space since $a ∉ τ$, and the only open superset of the closed set {c} is X which also contains a.

We introduce a famous Theorem.

**Urysohn Lemma.** Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map $f : X → [a,b]$ such that $f(x) = a$ for every $x$ in A, and $f(x) = b$ for every $x$ in B.

Urysohn Lemma implies that $T_3$-space is $T_{1\frac{1}{2}}$-space.

**Theorem 3.4** A $T_1$-space X is normal if and only if for each closed subset A of X and open set U containing A, there is an open set W containing A whose closure is contained in U.

**Proof.** Suppose that X is normal, and suppose that a closed set A and an open set U containing A. Let $B = X - U$, then B is a closed set. By
hypothesis, there exist disjoint open sets $V$ and $W$ containing $A$ and $B$, respectively. The set $\overline{V}$ is disjoint from $B$, since if $y \in B$, the set $W$ is a neighborhood of $y$ disjoint from $V$. Therefore, $\overline{V} \subseteq U$, as desired.

Conversely, let $A$, $B$ be disjoint closed sets in $X$. Let $U = X - B$. By hypothesis, there is an open set $V$ containing $A$ such that $\overline{V} = U$. The open sets $V$ and $X - \overline{V}$ are disjoint open sets containing $A$ and $B$, respectively. Thus $X$ is normal.

\textbf{Theorem 3.5}  \textit{Closed subspaces of a normal space are normal.}

\textit{Proof.} Let $Y$ be closed in $X$ and $A$ and $B$ be disjoint closed subsets of $Y$. Then $A$ and $B$ are disjoint closed subsets of $X$, and hence there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of $Y$ containing $A$ and $B$, respectively. Thus $Y$ is normal. \hfill \blacksquare

\textbf{Theorem 3.6}  \textit{The closed continuous image of a normal space is normal.}

\textit{Proof.} Suppose $X$ is normal and $f$ is a closed continuous map of $X$ onto $Y$. Let $A$ and $B$ be disjoint closed subsets of $Y$, then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of $X$ and we can find disjoint open sets $U_1$ and $U_2$ in $X$ containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. Since $f$ is closed, the sets $V_1 = Y - f(X - U_1)$ and $V_2 = Y - f(X - U_2)$ are open in $Y$. It is easily checked that $V_1$ and $V_2$ are disjoint and contain $A$.
and $B$, respectively. Thus $Y$ is normal.

\textbf{Theorem 3.7} Every metrizable space is normal.

\textit{Proof.} Let $X$ be a metrizable space with metric $d$. Let $A$ and $B$ be disjoint closed subsets of $X$. For each $a \in A$, choose $\epsilon_a$ so that the ball $B(a, \epsilon_a)$ does not intersect $B$. Similarly, for each $b \in B$, choose $\epsilon_b$ so that the ball $B(b, \epsilon_b)$ does not intersect $A$. Define

$$U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2}) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \frac{\epsilon_b}{2}).$$

Then $U$ and $V$ are open sets containing $A$ and $B$, respectively; we assert they are disjoint. For if $z \in U \cap V$, then

$$z \in B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2})$$

for some $a \in A$ and some $b \in B$. The triangle inequality applies to show that $d(a, b) < \frac{\epsilon_a + \epsilon_b}{2}$. If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$, so that the ball $B(b, \epsilon_b)$ contains the point $a$. If $\epsilon_a \geq \epsilon_b$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point $b$. Neither situation is possible.

\textbf{Theorem 3.8} Every compact Hausdorff space is normal.

\textit{Proof.} Let $X$ be a compact Hausdorff space. Then $X$ is regular. For if $x$ is a point of $X$ and $B$ is a closed set in $X$ not containing $x$, then $B$ is compact, and there exist disjoint open sets about $x$ and $B$, respectively.
Essentially the same argument as given in that lemma can be used to show that $X$ is normal: Given disjoint closed sets $A$ and $B$ in $X$, choose, for each point $a$ of $A$, disjoint open sets $U_a$ and $V_a$ containing $a$ and $B$, respectively. The collection $\{U_a\}$ covers $A$; because $A$ is compact, $A$ may be covered by finitely many sets $U_{a_1}, \ldots, U_{a_n}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_n} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_n}$$

are disjoint open sets containing $A$ and $B$, respectively. ■

**Theorem 3.9** Every well-ordered set $X$ is normal in the order topology.

It is, in fact, true that every order topology is normal; but we shall not have occasion to use this stronger result.

*Proof.* Let $X$ be a well-ordered set. We assert that every interval of the form $(x, y]$ is open in $X$: If $X$ has a largest element and $y$ is that element, $(x, y]$ is just a basis element about $y$. If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$, where $y'$ is the immediate successor of $y$.

Now let $A$ and $B$ disjoint closed sets in $X$; assume for the moment that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$. For each $a \in A$, there exist a basis element about $a$ disjoint from $B$; it contains some interval of the form $(x, a]$. (Here is where we use the fact that $a$ is not the smallest element of $X$.) Choose, for each $a \in A$, such an interval $(x_a, a]$ disjoint from $B$. Similarly, for each $b \in B$, choose an
interval \((y_b, b]\) disjoint from \(A\). The sets
\[
U = \bigcup_{a \in A} (x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B} (y_b, b]
\]
are open sets containing \(A\) and \(B\), respectively; we assert they are disjoint. For suppose that \(z \in U \cap V\). Then \(z \in (x_a, a] \cap (y_b, b]\) for some \(a \in A\) and some \(b \in B\). Assume that \(a < b\). Then if \(a \leq y_b\), the two intervals are disjoint, while if \(a > y_b\), we have \(a \in (y_b, b]\), contrary to the fact \((y_b, b]\) is disjoint from \(A\). A similar contradiction occurs if \(b < a\).

Finally, assume that \(A\) and \(B\) are disjoint closed sets in \(X\), and \(A\) contains the smallest element \(a_0\) of \(X\). The set \(\{a_0\}\) is both open and closed in \(X\). By the result of the preceding paragraph, there exist disjoint open sets \(U\) and \(V\) containing the closed sets \(A - \{a_0\}\) and \(B\), respectively. Then \(U \cup \{a_0\}\) and \(V\) are disjoint open sets containing \(A\) and \(B\), respectively. \(\blacksquare\)

**Theorem 3.10** Let \(X\) be a \(T_1\)-space. If for each pair \(A, B\) of disjoint closed sets in \(X\) there is a continuous function that separates \(A\) and \(B\), then \(X\) is normal.

**Proof.** Let \(A\) and \(B\) be disjoint closed subsets in \(X\), let \(f : X \to \mathbb{R}\) be a continuous function that separates \(A\) and \(B\). Thus \(f(A) = a\) and \(f(B) = b\) for some distinct real numbers \(a\) and \(b\). Since \(\mathbb{R}\) is Hausdorff, there exist disjoint open sets \(O_a\) and \(O_b\) containing \(a\) and \(b\), respectively. Then \(U = f^{-1}(O_a)\), \(V = f^{-1}(O_b)\) are disjoint open sets in \(X\) containing \(A\) and \(B\), respectively, so \(X\) is normal. \(\blacksquare\)
**Definition 3.11**  A space $X$ is said to be **completely normal** if for each pair $A$, $B$ of separated subsets of $X$ there exist disjoint open sets $U$ and $V$ such that $A$ is contained in $U$ and $B$ is contained in $V$.

**Theorem 3.12**  A space $X$ is completely normal if and only if every subspace of $X$ is normal.

*Proof.* Suppose $X$ is completely normal. Let $A$ be a subspace of $X$, and let $C$ and $D$ disjoint closed subsets of $A$. Then $\overline{C \cap D} = C \cap \overline{D} = \emptyset$. (If $x \in C \setminus \overline{C}$, then $x \not\in A$ and hence $x \not\in D$.) Therefore, there are disjoint open subsets $U$ and $V$ of $X$ such that $C \subset U$ and $D \subset V$. Hence $U \subset A$ and $V \subset A$ are disjoint open subsets of $A$ such that $C \subset (U \cap A)$ and $D \subset (V \cap A)$. Thus $A$ is normal.

Suppose $X$ is a topological space with the property that every subspace is normal. Let $A$ and $B$ be subsets of $X$ such that $A \cap \overline{B} = A \cap B = \emptyset$. Let $E = X \setminus (A \cap B)$. Then $A$ and $B$ are subsets of $E$. Let $C$ be the closure of $A$ in $E$ and let $D$ be the closure of $B$ in $E$. Then $C$ and $D$ are disjoint closed subsets of $E$. Since $E$ is normal, there exist disjoint open sets $U$ and $V$ such that $C \subset U$, $D \subset V$. Thus $X$ is completely normal. □
REFERENCES