X_n에서 Ricci 곡률 텐서 R_ij의 Conformal Change에 관하여

1. Introduction
2. Preliminaries
3. Conformal Change of the Ricci Tensor Rij
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1994年 2月

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主審________________

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국문要約

본 논문의 목적은 \( n \)차원 리만공간 \( X_n \)에서의 리즈곡률テン서의 Conformal change에 의한 변환을 연구하는 데 있다.

본 논문에서는 \( n \)차원 리만공간에서의 리즈곡률テン서의 conformal change에 의한 변환 \( R_y \)과 \( \overline{R_y} \) 사이의 관계를 \( g_y \)의 함수로 나타냈다.
ABSTRACT

The purpose of the present paper is to introduce the conformal change of
the Ricci tensor $R_{ij}$ in $n$-dimensional Riemannian space $X_n$. We derive the
change $R_{ij} \rightarrow \tilde{R}_{ij}$ of two $n$-dimensional Ricci curvature tensor introduced by
the conformal change in terms of $g_{ij}$.
Contents

Abstract ................................................................. i
Korean Abstract .................................................. ii
I. Introduction ...................................................... 1
II. Preliminaries ................................................... 3
III. Conformal change of the Ricci tensor $R_{ij}$ .... 6

References .......................................................... 13

247536
I. INTRODUCTION

Let $X_n$ be a Riemannian space based on metric defined by

\begin{equation}
(1.1) \quad ds^2 = g_{ij}dx^i dy^j \quad (i, j = 1, 2, \cdots, n).
\end{equation}

Where the coefficients $g_{ij}$ are functions of the coordinates $x^i$.

The quadratic differential form in the second member of (1.1) is called a Riemannian metric and a space which is characterized by such a metric is a Riemannian space.

Let $X_n$ be an $n$-dimensional Riemannian space defined by a fundamental real metric tensor $g_{ij}$ whose determinant is

\begin{equation}
(1.2) \quad g = \text{Det}((g_{ij})) \neq 0.
\end{equation}

By (1.2), there is a unique tensor $g^{ij} = g^{ji}$ defined by

\begin{equation}
(1.3) \quad g_{ij}g^{jk} = \delta^k_j.
\end{equation}

The tensor $g_{ij}$ and $g^{ij}$ will serve for raising and/or lowering indices of tensor quantities in $X_n$ in the usual manner.
Let $X_n$ be a generalized $n$-dimensional Riemannian space referred to a real coordinate system $x^i$, which obeys coordinate transformations $x^i \to \bar{x}^i$ for which

\begin{equation}
\text{Det} \left( \frac{\partial \bar{x}^i}{\partial x^j} \right) \neq 0.
\end{equation}

The purpose of the present paper is to introduce the conformal change of the Ricci curvature tensor $R_{ij}$ in $X_n$.

We derive the change $R_{ij} \to \bar{R}_{ij}$ of two $n$-dimensional Ricci curvature tensors introduced by the conformal change (3.1).
II. PRELIMINARIES

In this section, we introduce several concepts, notations and theorems in obtained by without proof ([1], [2], [3]).

These relations will be needed in our further considerations.

A) The metric tensor $g_{ij}$ is a symmetric tensor.

That is,

\begin{equation}
  g_{ij} = g_{ji}.
\end{equation}

B) The notations are defined by

\begin{align*}
  A_{(ij)} &= \frac{1}{2} (A_{ij} + A_{ji}) \\
  A_{[ij]} &= \frac{1}{2} (A_{ij} - A_{ji})
\end{align*}

C) The functions are defined by

\begin{equation}
  [k, ij] = \frac{1}{2} \left( \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right),
\end{equation}
and

\[(2.4)\quad \{ k \}_{ij} = g^{kk}[h, ij].\]

Where (2.3), (2.4) are called the Christoffel symbols of the first and second kind respectively.

D) The functions (2.3), (2.4) are symmetric with respect to the indices $i, j$.

That is,

\[(2.5a)\quad [k, ij] = [k, ji],\]

\[(2.5b)\quad \{ k \}_{ij} = \{ k \}_{ji}.\]

E) The tensors are defined by

\[(2.6)\quad R^a_{ijk} = \frac{\partial}{\partial x^j} \{ a \}_{ik} - \frac{\partial}{\partial x^k} \{ a \}_{ij} + \{ a \}_{bj} \{ b \}_{ik} - \{ a \}_{bk} \{ b \}_{ij},\]

and

\[(2.7)\quad R_{ij} = R^a_{ij a},\]

where (2.6), (2.7) are called the curvature tensor and the Ricci curvature tensor respectively.
F) Any other form of the Ricci curvature tensor is

\[
R_{ij} = \frac{\partial^2 \log \sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^a} \{ a \} \{ i j \} + \{ a \} \{ b \} \{ i a \} - \{ b \} \{ i j \} \frac{\partial}{\partial x^b} \log \sqrt{g}.
\]
III. Conformal change of the Ricci tensor $R_{ij}$

In this section, we investigate the change of the several functions induced by a conformal change of the tensor $g_{ij}$.

Consider two $n$-dimensional Riemannian space $X_n$, and $\overline{X}_n$. We say that $X_n$ and $\overline{X}_n$ are conformal if and only if

\begin{equation}
\overline{g}_{ij} = U^2 g_{ij},
\end{equation}

where $U = U(x)$ is an arbitrary functions of position with at least two derivatives.

**Agreement (3.1).** Throughout this section, we agree that, if $T$ is a function of $g_{ij}$, then we denote by $\overline{T}$ the same function of $\overline{g}_{ij}$. In particular, if $T$ is a tensor, so is $\overline{T}$.

**Theorem (3.1).** If $X_n$ and $\overline{X}_n$ are conformal, the following relation
holds:

(3.2) \[ \bar{g}^{ij} = U^{-2}g^{ij}. \]

**Proof.** In virtue of (1.3) and Agreement (3.1), we have (3.2). □

**Theorem (3.2).** The conformal change (3.1) induces the following change:

(3.3) \[ [ij, k] = U^2[ij, k] + (2U_i g_{j}k - U_k g_{ij})U, \]

where \( U_i = \frac{\partial U}{\partial x^i} \).

**Proof.** In virtue of (2.3) and Agreement (3.1), we have

(3.4) \[ [ij, k] = \frac{1}{2} \left[ \frac{\partial \bar{g}_{jk}}{\partial x^i} + \frac{\partial \bar{g}_{ik}}{\partial x^j} - \frac{\partial \bar{g}_{ij}}{\partial x^k} \right]. \]

Substituting (3.1) into (3.4),

\[ \frac{\partial \bar{g}_{jk}}{\partial x^i} = \frac{\partial}{\partial x^i}(U^2 g_{jk}) = 2U_i g_{jk} + U^2 \frac{\partial g_{jk}}{\partial x^i}. \]

Similarly,
\[ \frac{\partial g_{ik}}{\partial x^j} = 2UU_j g_{ik} + U^2 \frac{\partial g_{ik}}{\partial x^j}, \quad \cdots \quad (2) \]

(3.5)

\[ \frac{\partial g_{ij}}{\partial x^k} = 2UU_k g_{ij} + U^2 \frac{\partial g_{ij}}{\partial x^k}. \quad \cdots \quad (3) \]

Substituting (3.5) into (3.4), we have (3.3). 

THEOREM (3.3). The tensor \( \{ \frac{k}{ij} \} \) is transformed by the conformal change (3.1) as following:

(3.6)

\[
\left\{ \frac{k}{ij} \right\} = \left\{ \frac{k}{ij} \right\} + \frac{1}{U} (2U_i \delta^k_j - U_aj^{ak}g_{ij}).
\]

PROOF. In virtue of (3.1), Agreement (3.1), (2.2), (2.3), (2.4), and (3.3), the relation (3.6) may be derived as in the following way:

\[
\left\{ \frac{k}{ij} \right\} = \frac{a^{ak}[ij,a]}{U^2 g^{ak}[ij,a] + U(2U_i \delta^k_j - U_aj^{ak}g_{ij})} \\
= \frac{1}{U^2} g^{ak}[ij,a] + \frac{1}{U} (2U_i \delta^k_j - U_aj^{ak}g_{ij}) \\
= \left\{ \frac{k}{ij} \right\} + \frac{1}{U} (2U_i \delta^k_j - U_aj^{ak}g_{ij}).
\]
Remark (3.4). The quantity \( g \) is conformal invariant. That is

\[
\bar{g} = g.
\]

Proof. The relation (3.7) obtained by [1].

Theorem (3.5). The tensor \( B_{ij} \) is transformed by the conformal change (3.1) as following:

\[
\bar{B}_{ij} = B_{ij} + \left( \frac{1}{U} - \frac{1}{U^2} \right) C_{ij},
\]

where

\[
\bar{B}_{ij} = \frac{\partial}{\partial x^a} \left\{ \frac{a}{i_j} \right\},
\]

\[
C_{ij} = 2U_i U_j - U_a U_b g^{ab} g_{ij}.
\]

Proof. In virtue of (3.9) and Agreement (3.1), we have

\[
\bar{B}_{ij} = \frac{\partial}{\partial x^a} \left\{ \frac{a}{i_j} \right\}.
\]

Substituting (3.6) into (3.11), by using (3.9) and (3.10), we have

\[
\bar{B}_{ij} = \frac{\partial}{\partial x^a} \left\{ \left\{ \frac{a}{i_j} \right\} + \frac{1}{U} (2U_i \delta^a_j - U_b g^{ab} g_{ij}) \right\}
\]
\[ D_{ij} = D_{ij} + \frac{1}{U} (E_{ij} - F_{ij}) + \frac{1}{U^2} ((n + 2)U_i U_j - 2U_a U_b g^{ab} g_{ij}), \]

where

\[ E_{ij} = 2U_i \left\{ \begin{array}{c} \{ a \\ b_j \} \\ a_i \end{array} \right\}, \]

\[ F_{ij} = g^{\alpha a} U_\alpha. \]

**Proof.** In virtue of (3.1), Agreement (3.1), (2.2), (3.6), (3.13), (3.14) and (3.15), the relation (3.12) may be derived as in the following way:

\[ D_{ij} = D_{ij} \]

\[ = \left( \left\{ a \right\} + \frac{1}{U} (2U_i \delta^a_j - U_\alpha g^{\alpha a} g_{bj}) \right) \]
\begin{align*}
\times \left( \{ b \}_{ia} + \frac{1}{U} (2U_i \delta^b_a - U_a g^{ab} g_{ia}) \right) \\
= \{ a \}_{bij} \{ b \}_{ia} + \frac{1}{U} \left( 2U_i \{ a \}_{ij} \right) + 2U_a \{ a \}_{ij} \\
+ \frac{1}{U^2} (n + 2) U_i U_j - 2U_a U_b g^{ab} g_{ij} \\
= D_{ij} + \frac{1}{U} (E_{ij} - F_{ij}) + \frac{1}{U^2} ((n + 2) U_i U_j - 2U_a U_b g^{ab} g_{ij}).
\end{align*}

**Theorem (3.7).** The tensor $G_{ij}$ is transformed by the conformal change (3.1) as following:

(3.16) \[ \overline{G}_{ij} = G_{ij} + \frac{1}{U} H_{ij}, \]

where

(3.17) \[ G_{ij} = \{ b \}_{ij} \frac{\partial}{\partial x^b} \log \sqrt{g}, \]

(3.18) \[ H_{ij} = (2U_i \delta^b_j - U_a g^{ab} g_{ij}) \frac{\partial}{\partial x^b} \log \sqrt{g}. \]

**Proof.** In virtue of (3.17) and Agreement (3.1), we have

(3.19) \[ \overline{G}_{ij} = \overline{\{ b \}_{ij}} \frac{\partial}{\partial x^b} \log \sqrt{g}. \]

Substituting (3.6), (3.7), (3.18) into (3.19), we have (3.16).
Theorem (3.8). The Ricci curvature tensor $R_{ij}$ is transformed by the conformal change (3.1) as following:

$$\bar{R}_{ij} = R_{ij} + \frac{1}{U}(C_{ij} + E_{ij} - F_{ij} + H_{ij}) + \frac{1}{U^2}(nU_iU_j - U_aU_bg^{ab}g_{ij}).$$

Proof. In virtue of (2.8) and Agreement (3.1), we have

$$\bar{R}_{ij} = \frac{\partial^2 \log \sqrt{\bar{g}}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^a} \left\{ \frac{a}{ij} \right\} + \left\{ \frac{a}{ij} \right\} \left\{ \frac{b}{ia} \right\} - \left\{ \frac{b}{ij} \right\} \frac{\partial}{\partial x^b} \log \sqrt{\bar{g}}.$$  

Substituting (3.7), (3.8), (3.12) and (3.16) into (3.21), the relation (3.20) may be expressed by the following way:

$$\bar{R}_{ij} = \frac{\partial^2 \log \sqrt{\bar{g}}}{\partial x^i \partial x^j} - B_{ij} + \bar{D}_{ij} - \bar{G}_{ij} = R_{ij} + \frac{1}{U}(C_{ij} + E_{ij} - F_{ij} + H_{ij}) + \frac{1}{U^2}(nU_iU_j - U_aU_bg^{ab}g_{ij}).$$
REFERENCES


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