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Hadamard 공에 관한 研究

Hadamard Product

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1. Introduction

Let $\Delta$ be the unit disc in the complex plane $\mathbb{C}$. Denote by $H(\Delta)$ the set of analytic functions on $\Delta$. The Hadamard-product of $f(Z) = \sum_{n=0}^{\infty} a_n Z^n$, $g(Z) = \sum_{n=0}^{\infty} b_n Z^n$ in $H(\Delta)$ is

$$(f \ast g)(Z) = \sum_{n=0}^{\infty} a_n b_n Z^n$$

which is in $H(\Delta)[1]$. This product is associative, commutative, and distributive over addition. As an example

$$f \ast \frac{1}{1-SZ} = f(SZ) \quad (|S| \leq 1)$$

so that $1/(1-Z)$ is the identity for the product.

If $f$ is in the familiar class $S$ of normalized ($f(0) = f'$

$(0) - 1 = 0$) univalent functions in $\Delta$, then

$$f \ast \frac{Z}{1-SZ} = \frac{f(SZ)}{S} \quad (|S| \leq 1)$$

Consequently, if $f$ is in the set $K$ of convex functions of $S$, then

$$f \ast \frac{Z}{1-SZ} \in K$$
\[ g(Z) = \frac{1}{2\pi i} \oint_{|S| = \epsilon \leq 1} \frac{g(S)}{S-Z} \, ds \]

yields the relation

\[ (f \ast g)(z) = g(z) = \frac{1}{2\pi i} \int_{|S| = \epsilon} \frac{g(S)}{S-Z} \, ds \]

\[ = \frac{1}{2\pi i} \int_{|S| = \epsilon} f\left(\frac{z}{S}\right) g(s) \, \frac{ds}{S} \]  

(1)

In this thesis, we give first several properties of subclasses of S. Next, we characterized for these subclasses \[ [3]. \] the Hadamard Products.
2. Subclasses of S

Let
\[ P = \{ f \in H(\Delta) : \Re f > 0, \ f(0) = 1 \} \]

and let \( S_t \) be the set of starlike functions in \( S \).

**Proposition 1.** If \( f \in S \), then \( f \in S_t \) if and only if
\[ z f'/f \in P. \]

**Proof.** Note that \( f \in S_t \) if and only if every domain
\[ D_r = \{ f(z) : |z| < r < 1 \} \]

is starlike with respect to the origin. This comes from the Schwarz lemma.

Thus \( D_r \) is starlike if and only if \( \arg f( re^{i\theta} ) \) is a nondecreasing function of \( \theta \), that is
\[ 0 \leq \frac{\partial}{\partial \theta} \arg f( re^{i\theta} ) = T_m \frac{\partial}{\partial \theta} \log f( re^{i\theta} ) \]
\[ = \Re z f'/f \]

But \( \Re z f'/f > 0 \) by the maximum modulus theorem, since
\[ z f'/f |_{z=0} = 1 \]

Similarly, we also have the following result as in the previous proposition.
Proposition 2. If $f \in S$, then $f \in K$ if and only if $1 + zf''/f' \in P$.

Proposition 3. If $f \in S$, then $f \in K$ if and only if $zf' \in S$.

Proof. Let $g = zf'$. Then we have

$$\frac{zg'(z)}{gz} = \frac{zf''(z)}{f'(z)} + 1$$

Hence the result follows from propositions 1 and 2.

Suppose that $A$ is a subset of a linear space over $\mathbb{C}$ or $\mathbb{R}$. A point $z \in A$ is an extreme point of $A$ if $z \neq t \chi + (1-t)y$ for all distinct $\chi, y \in A$ and all $t \in (0,1)$.

We denote the set of extreme points of $A$ by $\text{ext } A$. If $A$ is a subset of a linear topological space, then the closed convex hull of $A$, denoted by $\overline{co} A$, is the smallest of all closed convex sets containing $A$.

We shall need the following theorem for the next proposition.

Theorem A [2, P.172]. Let $X$ and $Y$ be linear topological spaces and $L: X \to Y$ a linear homeomorphism.

If $A \subset X$, then $\overline{co} L(A) = L(\overline{co} A)$ and $\text{ext } (\overline{co} L(A)) = L(\text{ext } \overline{co} A)$.
Proposition 4.

\[ \text{CO}_K = \{ f \in S : \Re f/z > \frac{1}{2} \} . \]

Proof. The mapping \( L(g) = \frac{1}{2} z(g + g) \) is a linear homeomorphism of \( H(\Delta) \) onto the subspace \( \{ h \in H(\Delta) : h(0) = 0 \} \).

By the Theorem A, it maps the compact convex set \( P \) onto a compact convex set \( L(P) = \{ hf \in S : \Re f/z > \frac{1}{2} \} \).

Since \( \Re f/z > \frac{1}{2} \) for \( f \in K \), \( K \) is a subset of \( L(P) \). But \( K \) is closed, and hence \( K \) is a compact subset of the compact convex set \( L(P) \). Hence \( \text{CO}_K \subset L(P) \).

On the other hand, we know that \( \text{ext } P = \{(1+SZ)/(1-SZ) : |S|=1 \} \).

Thus \( L(\text{ext } P) = \{ Z/(1-SZ) : |S|=1 \} \). By Theorem A, \( \text{ext } (L(P)) \)

= \( L(\text{ext } P) \). Note that the maps \( Z/(1-SZ) \) belong to \( K \). Hence \( \text{ext } (L(P)) \subset K \) and \( L(P) = \text{CO}(\text{ext } L(P)) \subset K \) by the Krein-Milman theorem. Therefore \( L(P) = \text{CO}_K \).
3. Characterizations of Hadamard product

Lemma 1. If \( P(Z) = 1 + \sum_{n=1}^{\infty} p_n Z^n \) and \( q(Z) = 1 + \sum_{n=1}^{\infty} q_n Z^n \) are in \( P \), then

\[
\frac{1}{2} [ 1 + (P \ast q)(Z) ] = 1 + \frac{1}{2} \sum_{n=1}^{\infty} p_n q_n Z^n
\]

is also in \( P \).

Proof. Fix \( Z \in \Delta \). Then it follows from (1) that

\[
0 < \frac{1}{2\pi} \int_{1 \leq |s|} Re \{ \frac{Z}{S} \} Re \{ \frac{Z}{S} \} \frac{ds}{S} = \frac{1}{2} Re \left[ 1 + \sum_{n=1}^{\infty} p_n q_n Z^n + (P \ast q)(Z) \right]
\]

since \((P \ast q)(Z) = 1\).

Theorem 1. If \( f \) and \( g \) belong to \( \overline{COK} \), then \( f \ast g \) belongs to \( \overline{COK} \).

Proof. Let \( f(z) = Z + \sum_{n=2}^{\infty} a_n Z^n \) and \( g(z) = Z + \sum_{n=2}^{\infty} b_n Z^n \) be in \( \overline{COK} \).

Then

\[
2f(z)z^{-1} = 1 + 2 \sum_{n=1}^{\infty} a_{n+1} Z^n, \quad 2g(z)z^{-1} = 1 + 2 \sum_{n=1}^{\infty} b_{n+1} Z^n
\]

are in \( P \). Hence by lemma 1, we have
\[ 2(f * g)/z - 1 = 1 + 2 \sum_{n=1}^{\infty} a_{n+1} b_{n+1} Z^n \]
is in \( P \). Hence prop. 4 implies that \( f * g \in \overline{cK} \).

For the next result, we shall need the following lemma [2].

**Lemma 2.** If \( f \in K \), \( g \in S_t \), and \( F \in P \), then

\[ \operatorname{Re} \frac{f*(Fg)}{f*g} > 0 \]
in \( \Delta \).

**Theorem 2.** If \( f \) and \( g \) belong to \( S_t \), then \( f * g \) belongs to \( S_t \).

**Proof.** By prop. 1, we have that \( F = zg'/g \) is in \( P \).

Then

\[ \operatorname{Re} \frac{z(f*g)'}{f*g} = \operatorname{Re} \frac{f*(zg')}{f*g} = \operatorname{Re} \frac{f*(Fg)}{F*g} > 0 \]

by lemma 2. Hence prop. 1 gives that \( f * g \in S_t \).

**Theorem 3.** If \( f \) and \( g \) belong to \( K \), then \( f * g \) belongs to \( K \).

**Proof.** By prop. 3, \( zg' \in S_t \). Hence \( z(f*g)' = f*(zg') \in S_t \)

by Theorem 2. Consequently, \( f * g \in K \). by prop. 3.
REFERENCES


國文抄錄

単位円 \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) において、解析函数の集合を \( H(\Delta) \) とする。\( H(\Delta) \) の 2 函数 \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) の Hadamard コプル

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n
\]

として定義する。それらの convex 函数と星形函数の部分集合をそれぞれ \( K \) と \( S_t \) と呼ぶとき、

(a) \( f,g \in K \) ならば \( f \ast g \in K \),

(b) \( f,g \in S_t \) ならば \( f \ast g \in S_t \),

とする結果を得ている。