Flow instability in obstructed channel flow

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Flow instability in a channel obstructed by an infinite array of equi-spaced circular cylinders has been numerically studied. An immersed boundary method was employed to facilitate the placement of the cylinders within a Cartesian grid system. This flow configuration is relevant to a heat exchanger in which vortex generators play an important role in enhancing its heat-transfer capacity. The presence of the circular cylinders arranged periodically in the streamwise direction causes a significant topological change of the flow, leading to increasing susceptibility to flow instability. A parametric study has been carried out to investigate the effects of Reynolds number ($Re$) and the gap ($G$) between the cylinders and the channel wall not only on the primary instability for Hopf bifurcation, but also on the secondary instability leading to three-dimensional flow. The blockage ratio ($d/H$) and the distance between two neighboring cylinder centers were fixed as 0.2 and 3.333$H$, respectively, where $d$ and $H$ represent the cylinder diameter and the channel height, respectively. The Stuart-Landau equation was used to compute the growth rate of the primary instability. The characteristics of the primary instability, including the critical Reynolds numbers and the patterns of the subsequent unsteady flow, were identified. In particular, the crossover of flow topology from flow separation on the channel wall to separated free shear layers from the cylinders turned out to be the key point to explain the flow characteristics of an obstructed channel flow. In addition, the effects of $Re$ and $G$ on the flow-induced forces and the frequency of vortex shedding are also reported. A Floquet stability analysis was employed to study the secondary instability at a higher Reynolds number. It reveals the critical Reynolds number for the three-dimensional instability along with the most unstable spanwise wave number associated with each $G$ considered here. The dependency of the secondary instability on $G$ is also addressed.

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1. Introduction

To enhance heat transfer in a heat exchanger or on a gas-turbine blade of high temperature, engineers often attach vortex generators to such devices. The large-scale vortices induced by the vortex generators increase fluctuations of the flow, thus enhancing mixing and heat transfer [1]. Therefore, the issue of how to effectively disturb the flow is one of the key aspects to be considered in designing any device where enhancing heat transfer is desirable. Since the large-scale vortices are initiated by the primary instability in the separated free shear layers created by the vortex generators, and then these subsequently become three-dimensional (3D) due to the secondary instability, it is necessary to investigate the characteristics of the flow instabilities in depth in order to understand and control the routes to transition inside a heat exchanger. In the MHD literature, heat transfer has been considered in relation to the flow past a circular cylinder in a duct [2,3], which has applications in cooling blankets of fusion reactors, metallurgical processing, hydrodynamic generators, pumps, blood flow meters, to name a few.

Flow past a circular cylinder has been studied by many researchers [4–9], because it is the most simplest form of bluff-body flow that generates separated free shear layers. These free shear layers are susceptible to the primary instability leading to unsteady flow. It has been revealed that periodic vortex shedding occurs at $Re_{c1} = 46$ [10]; here the Reynolds number is based on the free stream velocity and the cylinder diameter, and the subscript $c1$ indicates the critical Reynolds number for the primary instability. For higher $Re$, 3D instability sets in; mode A at $Re_{c2} = 188.5 \pm 1$, and mode B at $Re_{c2} = 259 \pm 2$ [10]. Here, the subscript $c2$ indicates the critical Reynolds number for the 3D instability. Mode A occurs with a dominant spanwise wavelength of approximately three to four cylinder diameters, manifesting as a spanwise distortion of the Karman vortices shed from the cylinder. A pair of counter-rotating streamwise vortices is alternately and periodically formed in the upper region and in the lower region of the cylinder wake. The sense of those pairs is opposite, which is called “odd reflection-translation symmetry” [10] of mode A. On the other hand, mode B appears with a rather short spanwise

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wavelength of about one diameter [10], and the pairs of counter-rotating streamwise vortices exhibit “even reflection-translation symmetry”.

In other flow configurations such as flow past a square cylinder with an incidence angle less than approximately 3° [11–13] and flow past a ring [14] or an airfoil [15], a quasi-periodic mode (QP) is observed. Mode QP has a complex-conjugate pair of eigenvalues. An imaginary component implies a quasi-periodic instability. In the case of the flow past a square cylinder, \( Re_{c2} \) of mode QP is higher than those of modes A and B [16].

Sheard et al. [17] investigated the 3D instability in the flow past toroidal bluff bodies, and found mode C in addition to modes A and B. Mode C differs from modes A and B in that it is a subharmonic instability, meaning that the associated Floquet multiplier eigenvalue exceeds the unit circle on the negative real axis rather than the positive real axis for the synchronous A and B instabilities [14]. Mode C is excluded from the possible instability branches when the base flow has a half-period reflective symmetry [18]. Instead, mode C is destabilized when the symmetry is broken, for instance, flow past slender bluff rings [17], flow past an inclined square cylinder [11–13], and flow past two circular cylinders in a staggered arrangement [19]. Mode QP appears to be replaced by mode C when the half-period reflective symmetry is broken. In the flow past a square cylinder, for example, mode QP disappears and mode C emerges with increasing incidence angle [11–13].

These cylinders can be simple and efficient vortex generators that can be utilized for heat transfer enhancement in a heat exchanger. However, their instability characteristics will be substantially altered if the cylindrical vortex generators are confined in an internal flow.

In this study, we consider low-\( Re \) laminar flow in a channel with an infinite streamwise array of equispaced identical circular cylinders as a model flow configuration representing a micro-channel or an internal heat exchanger with cylindrical vortex generators (Fig. 1). Both the primary and the secondary instabilities, associated with the current flow configuration, are responsible for the increase of fluctuation [20–23]. The primary instability is often referred to as a Hopf bifurcation in which a steady flow bifurcates into a time-periodic flow, whereas the secondary instability leads to a 3D flow. In particular, we focus on the effects of Reynolds number (\( Re \)) and the gap (\( G \)) between the cylinders and the channel wall on the primary and secondary instabilities with a fixed blockage ratio (\( d/H = 0.2 \)) and a fixed streamwise distance (\( L = 3.333H \)) between two neighboring cylinder centers. These values of the geometrical parameters are the same as those in the previous study of Schatz et al. [24]. Since the flow-induced force acting on the cylinders can cause structural vibration and noise, it is also important to know its dependency on the flow parameters. Thus the vortex-shedding frequency and the force coefficients on each cylinder were computed in the laminar range of \( Re \) with various values of \( G \). The results reported here shed light on understanding and controlling laminar–turbulent transition in the internal heat exchangers with vortex generators, which can be useful in enhancing heat transfer.

2. Formulation and numerical methods

2.1. Base flow

The governing equations for incompressible flow, modified for an immersed boundary method [25], are as follows:

\[
\nabla \cdot \mathbf{u} = q = 0
\]

\[
\frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot (\mathbf{uu}) - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} + \mathbf{f}
\]

where \( \mathbf{u}, p, q, \) and \( \mathbf{f} \) represent velocity vector, pressure, mass source/sink, and momentum forcing, respectively. All the physical variables except for pressure are nondimensionalized by the half channel height (\( h \)) and the centerline velocity of the channel flow without the cylinders (\( U_{cl} = 3Q/4h \), here \( Q \) denotes the flow rate). Pressure is nondimensionalized by a reference pressure (\( P_{ref} \)) and the dynamic pressure. We fixed the flow rate in time by following You et al. [26]. The governing equations are discretized by a finite-volume method on a nonuniform staggered Cartesian grid system (Fig. 2). Spatial discretization is second-order accurate. A hybrid scheme is used for time advancement; the nonlinear terms and the forcing terms are explicitly advanced by a third-order Runge-Kutta scheme, and the diffusion terms are implicitly advanced by the Crank-Nicolson method. A fractional step method [27] is employed to decouple the continuity and momentum equations. The Poisson equation resulted from the second stage of the fractional step method is solved by a multigrid method. For detailed description of the numerical method used in the current investigation, see Yang and Ferziger [28].

Two-dimensional (2D) base flow was computed with the following boundary conditions. A no-slip condition is imposed on the cylinder surface and on the channel walls, while the flow is assumed to be periodic in the streamwise direction. The computational domain is defined as \( 0 \leq x \leq 6.666 \) and \( 0 \leq y \leq 2 \), and contains a single cylinder positioned with its center a distance 1.5\( h \) downstream of the inlet (Fig. 2).

The number of cells used was \( 272 \times 240 \) in the streamwise (\( x \)) and the vertical (\( y \)) directions, respectively. For the circular cylinder, 48 \( \times \) 48 cells were allocated regardless of \( G \). To ensure grid-independency, we performed a grid-refinement study with \( G/d = 0.75 \), \( Re = 230 \) by doubling the numerical resolution in each direction (\( 544 \times 480 \)), and found that the fine-grid simulation yields very little difference compared to the coarse-grid one, for example, 0.95% in mean lift coefficient, 0.42% in its rms (root-mean-square) fluctuation, and 0.11% in Strouhal number (\( St \)) of vortex shedding.

2.2. Floquet stability analysis

The onset of the secondary instability leading to 3D flow can be detected by a Floquet stability analysis in which an instantaneous velocity field is decomposed into a 2D base flow with a period \( T \)

![Fig. 1. Flow configuration.](image)

![Fig. 2. Computational domain and boundary conditions.](image)
\[ \mathbf{u}(x,y,z,t) = \mathbf{U}(x,y,z) + \mathbf{u}'(x,y,z,t). \] (3)

Substituting Eq. (3) and similar expressions for the other variables into the Navier–Stokes and continuity equations, and then linearizing them, one can obtain the following governing equations for the perturbation velocity field,

\[ \nabla \cdot \mathbf{u}' - q' = 0 \] (4)

Here, the additional terms for the immersed boundary method are also included. The perturbation velocity and pressure fluctuations are assumed to be periodic in the streamwise direction, and the no-slip condition is imposed on all the solid surfaces.

By defining the operator \( \mathbf{L} \) so that \( \mathbf{L}(\mathbf{u}') \) is the right-hand side of the linearized equation, Eq. (5) can be written symbolically as

\[ \frac{\partial \mathbf{u}'}{\partial t} = -\nabla \cdot (\mathbf{u}' \mathbf{U} + \mathbf{u}' \mathbf{U}) - \nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' + \mathbf{f}. \] (5)

In this study, we use the power-type method in conjunction with an immersed boundary method [25] to calculate the Floquet instability of the time-periodic waves past the circular cylinders periodically located in tandem. For the sake of convenience, the term “Floquet multiplier” implies the one that has the maximum magnitude among all the Floquet multipliers from now on, and the subscript, “max”, is dropped.

Eqs. (4) and (5) were temporally and spatially discretized in the same way as for the base flow (see Section 2.1). The 2D time-periodic base flows were first computed with 272 × 240 grid cells in \( x \) and \( y \) directions, respectively; 32 snapshots were saved for one period of vortex shedding or vortex oscillation. They were fed to Eq. (5), being Fourier interpolated at each time step. For the Floquet stability analysis, the same numerical resolution (272 × 240 cells in \( x \) and \( y \) directions) was used. Doubling the number of snapshots at \( G/d = 0.75, Re = 230, \beta = 2.0 \) reveals a difference of 0.031% in the growth rate of the eigenmode compared to the 32-snapshot result, confirming the adequacy of 32 snapshots.

3. Results and discussion

The codes used in the current investigation, namely the base-flow solver and the perturbation-field solver, were validated for the flow past a circular cylinder and for the flow past an inclined square cylinder, respectively. One can refer to our previous publications [31–33].

3.1. Steady flow

Fig. 3 presents the streamwise velocity profiles at \( x = 1.5, 2.5, 6.0 \) for the various cases of \( G/d \) at \( Re = 120 \); the flow is steady at this Reynolds number regardless of \( G \). At \( x = 1.5 \) (Fig. 3a) where the cylinder is located, the gap flow between the cylinder and the lower channel wall drastically increases as \( G/d \) increases. Across the channel a parabolic profile is quickly being approached downstream of the cylinder (Fig. 3c), especially when the cylinder is close to the channel wall.

Steady streamlines at \( Re = 120 \) are shown in Fig. 4 for the various cases of \( G/d \). When the gap is narrow, the flow accelerates and then decelerates between the cylinder and the lower channel wall, resulting in flow separation and a recirculation region on the channel wall downstream of the cylinder (Fig. 4a and 4b), see also Fig. 3b. The recirculation region becomes small and eventually vanishes as the gap increases (Fig. 4c). With a larger gap, the cylinder is situated in a region of higher streamwise velocity, causing the “local” Reynolds number to increase. Consequently, flow separates on the cylinder surface, and form a vortex or a pair of
counter-rotating vortices depending upon $G$ right behind of the cylinder (Fig. 4d–h). Fig. 4 demonstrates that flow topology can completely change depending on how close to the channel wall the cylinder is positioned even for a fixed $Re$. This implies that $G$ should be a key parameter in the flow instability.

3.2. Primary instability

Presence of a cylinder in channel flow makes the flow more susceptible to the flow instability leading to an unsteady flow (hereafter, called “primary instability”). Furthermore, as pointed out in Section 3.1, the gap between the cylinder and the channel wall significantly affects the flow topology, leading to a drastic change of the primary instability. One can study the instability by using the Stuart-Landau (SL) equation. Near the critical Reynolds number of the primary instability ($Re_{c1}$), the following SL equation holds [34–35]:

$$\frac{dA}{dt} = \sigma_i(Re)A - |A|^2A + \cdots$$  \hspace{1cm} (8)

Here, $A(t)$ represents a characteristic complex amplitude, while $\sigma = \sigma_i + i\sigma_r$ and $l = l_i + il_r$ denote the linear growth rate and the first Landau constant, respectively. After linearizing the real part of Eq. (8) near $Re_{c1}$, one can obtain the following equation for the linear growth rate of the least stable mode [34]:

$$\frac{1}{|A|} \frac{dA}{dt} = \sigma_i - l_i|A|^2 = \sigma_i\left(1 - \frac{|A|^2}{|A|_{sat}^2}\right)$$  \hspace{1cm} (9)

where $|A|_{sat}$ represents the envelope amplitude in the saturated state. Also note that $\sigma_i$ is proportional to $Re - Re_{c1}$ in the vicinity of $Re_{c1}$ as follows:

$$\sigma_i = K(Re - Re_{c1})$$  \hspace{1cm} (10)

Here, $K$ is a positive constant. Eq. (10) is valid in the range of $-10 \leq Re - Re_{c1} \leq 20$ [34]. Park [36] and Sohankar et al. [37] chose the envelope of the lift coefficient to compute $A(t)$, and we also did so. Schatz et al. [24] experimentally and numerically obtained the critical Reynolds number of the primary instability for the same flow configuration as ours with $G/d = 0.75$. Fig. 5 presents the current result of the grow rate of the primary instability ($\sigma_i$), together with the experimental and numerical results of Schatz et al. [24] for comparison. The subcritical values of $\sigma_i$ were obtained by initially imposing random noise of 5% of $U_{cl}$ in magnitude on the steady 2D flow field, and then computing the exponential decay of the lift coefficient envelope. The current result ($Re_{c1} = 135.5$) is in good agreement with the numerical stability analysis of Schatz et al. [24] ($Re_{c1} = 134$).

As seen in Fig. 4, $G$ significantly influences the flow topology, consequently affecting the primary instability. In Fig. 6, the critical Reynolds numbers for various values of $G/d$ are shown. The appearance of overlapping local minima (two clear ones at $G/d = 0.75$, $G/d = 1.375$) implies that for different $G/d$ there are different mechanisms producing the onset of unsteady flow. These mechanisms work best at the local minimum location, and less effectively away from the minimum. In Fig. 6, the entire domain has been divided into 4 regimes bounded by the $Re_{c1}$ curve and the two gray bars, and then each regime has been properly named based on its key feature, namely, Obstacle near the channel wall, Vortex shedding from the cylinder, Recirculation on the channel wall, and Recirculation bubble behind the cylinder, respectively. In the regime of “Obstacle near the channel wall”, the cylinders are close to the channel wall, thus act as obstacles on the channel wall. In the
regime of “Vortex shedding from the cylinder”, unsteady flow develops in the traditional fashion for flow past a cylinder. The steady regimes are similarly classified into “Recirculation on the channel wall” and “Recirculation bubble behind the cylinder”.

Comparing with the critical Reynolds number of the plane channel flow based on a linear stability analysis ($Re_{c1} \approx 5770$, Orszag [38], White [39]), one can notice drastic increase of flow instability (i.e. low $Re_{c1}$) with the presence of the cylinder. It is also seen in Fig. 6 that the flow is stabilized as the cylinder gets closer to the channel wall. Since the separated free shear layers from the cylinder surface are suppressed near the channel wall (Fig. 4a and b), one can conjecture that the instability associated with the separated free shear layers (Fig. 4c-h) would be the dominant one in the current flow configuration as indicated by low $Re_{c1}$ for $G/d \geq 0.75$.

3.3. Unsteady characteristics

In Fig. 7, are shown instantaneous streamlines at some selected instants for a supercritical Reynolds number near $Re_{c1}$. When the cylinder is positioned close to the channel wall ($G/d = 0.25$, Fig. 7a), the local flow velocity is low around the cylinder, thus no flow separation occurs on the cylinder surface. Instead, flow separates on the channel wall due to the adverse pressure gradient induced by the gap flow, and forms a recirculation region that travels downstream and splits into two afterwards. They enter the inlet owing to the periodic boundary condition, and then vanish as the cylinder is approached. The re-entry of the travelling recirculation region causes an oscillation of the upstream stagnation point on the cylinder surface, significantly affecting the flow-induced forces (lift and drag) on the cylinder. With $G/d = 0.5$, the unsteady characteristics are similar to those of the previous case of $G/d = 0.25$, except that splitting of the recirculation region does not occur, and its size is reduced (Fig. 7b). In the case of $G/d = 0.75$, one can see only a very small recirculation region travel downstream along the channel wall, and disappear without the re-entry (Fig. 7c). However, one can also identify flow separation on the cylinder surface, even though it is weak, and does not alternate under the influence of the channel wall (Fig. 7c). At a higher $Re$ ($=155$) with the $G/d$ unchanged, the recirculation region intensifies and re-enters the inlet, while the flow separation now alternates at the upper and lower parts of the cylinder surface (Fig. 7d). This certainly confirms that the higher $Re$ is, the more the flow instability intensifies. With $G/d = 1.0$, the recirculation region on the channel wall is completely eliminated, and alternating undulation of the separated free shear layer is seen (Fig. 7e). Fig. 7f presents instantaneous streamlines for $G/d = 1.375$, $Re = 130$. One can recall that the second local minimum of $Re_{c1}$ occurs at $G/d = 1.375$ (Fig. 6).

The flow pattern is similar to that of $G/d = 1.00$, $Re = 150$ (Fig. 7e). With a larger gap ($G/d = 2.0$), alternating vortex shedding, which is a typical feature of the shear layer instability in an open domain, is clearly seen (Fig. 7g). The crossover of flow topology, from a wall-layer separation to a bluff-body separation, is noticed in an approximate overlap range of $0.875 < G/d < 1.0$, and the latter is more susceptible to the primary instability (Fig. 6). The crossover was consistently noticed in the subcritical steady flows (Fig. 4).

3.4. Flow-induced forces

The interference between the recirculation region on the channel wall and the separated free shear layer behind the cylinder also affects the flow-induced forces on the cylinder. The drag and lift coefficients are defined as

$$C_D = \frac{D}{(1/2) \rho U_D^2 d}$$

$$C_L = \frac{L}{(1/2) \rho U_D^2 d}$$

They were computed for the selected sets of $(G/d, Re)$ as indicated in Fig. 8. Here, $D$ and $L$ denote drag and lift forces per unit span, respectively. The hollow symbols represent cases where the 2D base flow is steady or periodic in time with a single frequency, while the solid symbols denote cases where multiple frequencies are dominant in the 2D base flow. The Floquet stability analysis (which will be discussed later) was also performed on the hollow-symbol cases only. Some examples of $C_D$ variation are shown in Fig. 9. Time histories of $C_D$ along with instantaneous vorticity contours for $G/d = 1.75$, and $Re = 160, 175, 185$ are presented in Fig. 9. Here, the vorticity contours were taken at an instant of maximum $C_D$. When $Re = 160$ (Fig. 9a), one specific frequency ($St = 0.362$) is prevailing over the others. However, multiple frequencies are dominant at $Re = 175$ (Fig. 9b) as indicated by the coexistence of high-frequency oscillation ($St = 0.444$) and low-frequency modulation ($St = 0.358$). In the flow at $Re = 185$ (Fig. 9c), the modulation disappears and one single frequency ($St = 0.444$) is again dominant over the others. The vorticity contours of $G/d = 1.75$, $Re = 160$ (Fig. 9a) reveal a longer length scale, whereas that of $G/d = 1.75$, $Re = 185$ (Fig. 9c) is shorter than the others. The case of $G/d = 1.75$, $Re = 175$ is in between. The length scales are consistent with the frequencies observed.

Fig. 10 presents the time-averaged force coefficients and their fluctuation rms (root-mean-square) values for the selected cases of $(G/d, Re)$. Fig. 10a shows the time-averaged drag coefficient ($C_D$). When the cylinder is positioned close to the channel wall (i.e. low $G/d$), $C_D$ yields low values due to the retarded local flow velocity. The opposite is the case for large $G/d$. For each $Re$
investigated, the time-averaged drag coefficient is maximum when the cylinder is at the center of the channel (i.e. $G/d = 2.0$). For each $G/d$ investigated, $C_D$ tends to monotonically decrease with increasing $Re$ in the subcritical range, consistent with other bluff-body flows [33]. The $C_D$ tendency in the supercritical range, however, depends on $G/d$. In the cases of large $G/d$ (for example, $G/d = 2.0$), the
The $\overline{C_D}$ curve turns up at the critical Reynolds number, and then monotonically increases afterwards. On the other hand, $\overline{C_D}$ still decreases even in the range of $Re > Re_{c1}$ when $G/d$ is equal to or less than 1.0. Nevertheless, its slope is milder than that of the subcritical range.

This dependency of the $\overline{C_D}$ slope on $G/d$ is strongly correlated with the crossover of flow topology discussed in Section 3.3. With $G/d \geq 1.0$, the instability associated with the separated shear layer prevails without any recirculation region on the channel wall (Fig. 7). The disturbed wake flow re-enters the inlet due to the streamwise periodicity, and then increases the local velocity upstream of the cylinder, resulting in a significant increase of drag force. To quantify this notion, we define $u_{cyl}$ as follows:

$$u_{cyl} = \left( \int_G^{G+d} u(1.5h - d, y, t) dy \right) / d. \quad (13)$$

Here, $u_{cyl}$ denotes the spatially averaged streamwise velocity on the vertical span of 1.0$d$, one cylinder diameter upstream of the cylinder center (Fig. 1). In Table 1, the time-averaged $u_{cyl}$ at the selected Reynolds numbers is listed for $G/d = 2.0$ and 0.625. In the case of $G/d = 2.0$, the critical Reynolds number of the primary instability is $Re_{c1} = 136.4$ above which vortices are shed in the cylinder wake. It is seen in Table 1 that $u_{cyl}$ is almost constant in the subcritical range of $Re$, but substantially increases beyond $Re_{c1}$. This means that the streamwise local mean velocity just upstream of the cylinder increases more than it would normally do with increasing $Re$, if vortices are shed in the cylinder wake. On the contrary, $u_{cyl}$ is almost constant in the entire range of $Re$, including both the subcritical and supercritical ranges, if there is no vortex shedding ($G/d = 0.625$). Note that $Re_{c1} = 142.3$ for $G/d = 0.625$ (Fig. 6). Since there are recirculation regions on the channel wall
for $G/d = 0.625$, one can conclude that the re-entering recirculation regions contribute little to the increase of drag force, whereas the re-entering vortices contribute significantly.

Fig. 10b presents the time-averaged lift coefficient ($C_l$) for all the computed sets of $(G/d, Re)$. In the case of $G/d = 2.0$, $C_l$ is zero regardless of $Re$ because of the perfect geometrical symmetry with respect to the centerline of the channel. It is also seen in Fig. 10b that for $G/d \leq 0.625$, $C_l$ is positive in the entire $Re$ range considered, but negative for $G/d \geq 0.875$. In the specific case of $G/d = 0.75$, however, $C_l$ is positive for lower $Re$, but becomes negative for higher $Re$. This can be explained by the pressure distribution on the cylinder surface. Fig. 11 shows the time-averaged pressure coefficient ($C_p$) along the cylinder surface for $G/d = 0.75$, $Re = 120$ (subcritical) and for $G/d = 0.75$, $Re = 150$ (supercritical). The pressure coefficient is defined as $C_p = \frac{p - p_{pref}}{1/2 \rho U^2}$, where $p_{pref}$ denotes the spatially-averaged pressure at $x = 0$. In the case of $Re = 120$ (steady), the pressure on the lower part of the cylinder ($60^\circ \leq \theta \leq 137^\circ$) is higher than that on the upper part of the cylinder (Fig. 11a) due to the blockage effect between the cylinder and the channel wall, and the upstream stagnation point is near the centerline ($\theta \approx 13^\circ$). Therefore, there is a strong upward force exerted on the lower part of the cylinder, but a weak downward force component in the vicinity of the stagnation point, resulting in a net positive lift force. In the case of $Re = 150$, however, the upstream stagnation point considerably shifts towards the upper part of the cylinder ($\theta \approx 22^\circ$, Fig. 11b) because the re-entering recirculation region distorts the flow path as it approaches the cylinder (Fig. 7d). Consequently, the strong downward force component in the vicinity of the stagnation point considerably offsets the upward force on the lower part of the cylinder, resulting in a negative $C_l$. 

Fig. 12 shows $C_p$ along the cylinder surface for $Re = 120$ and $G/d = 0.5, 1.0, 1.5, 2.0$, respectively. At this $Re$, flow is steady regardless of $G/d$ (Fig. 6). When the cylinder is positioned at the center of the channel ($G/d = 2.0$), the distribution of $C_p$ is perfectly symmetric with respect to the channel centerline, and the upstream stagnation point is located at $\theta = 0^\circ$. As the cylinder gets closer to the channel wall, the asymmetry of the $C_p$ distribution becomes more pronounced such that the pressure distribution on the lower half of the cylinder becomes overall higher than that on the upper half, contributing to the positive lift force. On the other hand, the upstream stagnation point shifts towards the upper part of the cylinder (Figs. 4 and 12) for $G/d = 1.5 \ (\theta \approx 8^\circ)$ and 1.0 ($\theta \approx 13^\circ$), contributing to the negative lift force component, then returns towards $\theta \approx 8^\circ$ for $G/d = 0.5$. The net effect from the two turns out to be negative for $G/d = 1.5$ and 1.0, and positive for $G/d = 0.5$ (Fig. 10b).

The rms of drag coefficient fluctuation ($C_{D_{rms}}$) is presented for all the computed sets of $(G/d, Re)$ in Fig. 10c. For the Reynolds numbers below $Re_{cr}$, the flow is steady, thus $C_{D_{rms}} = 0$. Overall, the slope of $C_{D_{rms}}$ is steep for $G/d \leq 0.875$ where the instability associated with the recirculation regions on the channel wall prevails, while the slope of $C_{D_{rms}}$ is mild for $G/d = 1.0$ where the vortex shedding in the cylinder wake is dominant. This implies that the recirculation regions on the channel wall strongly correlate with
$C_{D,\text{rms}}$ more than the shed vortices do. In Fig. 13, we plotted phase diagrams by using instantaneous values of $u_{\text{cyl}}$ and $C_D$ for various values of $Re$ and $G/d$. Steady flows are represented by solid squares, and the cases with multiple shedding frequencies (represented by solid circles in Fig. 8) are not shown for clarity. It is seen that $u_{\text{cyl}}$ and $C_D$ are highly correlated and their fluctuations are large when the gap is small ($G/d = 0.75$), whereas the opposite is the case when the gap is large ($G/d = 2.0$). Since the fluctuation of $u_{\text{cyl}}$ for $G/d = 0.75$ is induced by the recirculation regions travelling along the channel wall, one can say that the travelling recirculation regions on the channel wall are responsible for the high $C_{D,\text{rms}}$. Nevertheless, the larger $G/d$ is, the higher mean drag is obtained, as expected.

To examine the correlation between $u_{\text{cyl}}$ fluctuation ($u_{\text{cyl}}$) and $C_D$ fluctuation ($C_D$) more closely, we introduce a correlation coefficient defined as

$$C(C_D, u_{\text{cyl}}) = \frac{\bar{C_D} \bar{u}_{\text{cyl}}}{\left(\bar{C_D}^2 \bar{u}_{\text{cyl}}^2\right)^{1/2}}.$$  \hspace{1cm} (14)

Here, the overbar represents time averaging. In Fig. 14, the correlation coefficient is plotted for the same cases shown in Fig. 13. The correlation is very strong when $G/d = 0.75$, but that is not the case when the gap is large ($G/d = 2.0$). The case with $G/d = 1.375$ is situated between the two as in Fig. 14. It is also noted that the abrupt change of the correlation coefficient for $G/d = 2.0$ reflects the topological change of the base flow with increasing $Re$ (see Figs. 8 and 9).

The rms of lift coefficient fluctuation ($C_{L,\text{rms}}$) is presented for all the computed sets of $(G/d, Re)$ in Fig. 10d. For the Reynolds numbers below $Re_{ct}$, the flow is steady, thus $C_{L,\text{rms}} = 0$. Overall, the slope of $C_{L,\text{rms}}$ is steep for $G/d > 1.0$ where the instability associated with the vortex shedding in the cylinder wake prevails, while the

<table>
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<tr>
<th>$G/d$</th>
<th>$Re$</th>
<th>$u_{\text{cyl}}$</th>
<th>$Re$</th>
<th>$u_{\text{cyl}}$</th>
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Table 1
Time-averaged $u_{\text{cyl}}$ at the selected Reynolds numbers.
slope of $C_{L,m}$ is mild for $G/d \leq 0.875$ where the instability associated with the recirculation regions travelling on the channel wall is dominant. This implies that the vortex shedding in the cylinder wake strongly correlates with $C_{L,m}$ more than the travelling recirculation regions do. One can recall that the opposite was the case for $C_{D,m}$ (Fig. 10c). Fig. 15 presents distributions of the pressure coefficient around the circular cylinder at the instants of the maximum (solid line) and minimum (dotted line) lift, respectively, for $Re = 150$ and $G/d = 0.75, 1.75$. In the case of $Re = 150$, $G/d = 0.75$ (Fig. 15a), the pressure in the vicinity of the lower portion of the cylinder surface ($\theta \approx 270^\circ$) is higher than that of the upper portion ($\theta \approx 90^\circ$) at both instants due to the blockage effect, resulting in a small $C_L$ in magnitude. On the other hand, the case of $Re = 150$, $G/d = 1.75$ (Fig. 15b) reveals that the pressure near the lower portion of the cylinder ($\theta \approx 270^\circ$) is higher than that of the upper portion ($\theta \approx 90^\circ$) when $C_L$ is maximum, while the opposite is the case when $C_L$ is minimum. This alternating pressure characteristic is due to the vortex shedding in the cylinder wake, and yields a large $C_L$ in magnitude.

3.5. Dominant Strouhal number of the base flow

Fig. 16 presents the dominant Strouhal number ($St$) for all the cases considered here, computed by using the time-history of $C_L$ on the cylinder. In the range of $G/d \leq 0.875$, the dominant $St$ is determined by that of the recirculation regions travelling on the channel wall because vortex shedding does not occur behind the
cylinder. For \( G/d \geq 1.0 \), however, the vortex shedding in the cylinder wake dominates the flow, resulting in a drastic increase of the frequency. In particular, there exists a “frequency crossover” for each \( G/d \) that belongs to \( G/d \geq 1.625 \). For each \( Re \), the frequency increases with increasing \( G/d \), while the St is almost constant for a fixed \( G/d \), excluding the crossover.

3.6. Onset of the secondary instability

The instability through which a time-periodic 2D base flow bifurcates into a 3D flow is called the secondary instability or the three-dimensional instability. In Floquet stability analysis, the onset of the secondary instability is determined by the magnitude of the Floquet multiplier described in detail in Section 2.2. If one can find a Floquet multiplier of which magnitude is larger than 1.0 for a given time-periodic 2D base flow, it can be said that the base flow is unstable to 3D disturbances, and the flow eventually becomes three-dimensional (see Section 2.2). The fluctuating velocity field \( \mathbf{u}_f(x, y, z, t) \) and its vorticity field \( \omega_f(x, y, z, t) \) corresponding to the Floquet mode of a given spanwise wave number \( \beta \) can be written as follows:

\[
\mathbf{u}_f(x, y, z, t) = (\bar{u}\cos \beta z, \bar{v}\cos \beta z, \bar{w}\sin \beta z) \tag{15}
\]

\[
\omega_f(x, y, z, t) = (\bar{\omega}_x \sin \beta z, \bar{\omega}_y \sin \beta z, \bar{\omega}_z \cos \beta z). \tag{16}
\]

In Fig. 17, the current results of \(|\mu|\) are presented in a range of \( \beta \) for \( G/d = 0.75 \), and \( Re = 150, 155, 160 \), revealing good agreement with the numerical results of Schatz et al. [24]. The most unstable \( \beta \) is 0.8, and independent of Reynolds number in the \( Re \) range shown. Fig. 18 presents the critical Reynolds number of the secondary instability \( (Re_{c2}) \) for each \( G/d \) under consideration together with the critical Reynolds number of the primary instability (Hopf bifurcation, \( Re_{c1} \)). Three distinct modes are identified, namely, mode C, mode QP (QP stands for “Quasi-Periodic”), and mode I. Mode C is characterized by the doubled time period of the Floquet mode compared with that of the base flow, while mode QP has a time period that is different from that of the base flow[30]. Furthermore, mode C reveals a converged \(|\mu|\) obtained by a power-type method, whereas mode QP demonstrates an oscillating \(|\mu|\). As an example of mode C, Fig. 19 shows time evolution of the streamwise vorticity component of the Floquet mode \( (\bar{\omega}_x) \) taken at \( x = 3.5 \) for \( Re = 240, G/d = 0.25, \beta = 1.2 \). The light and dark colours represent positive and negative values of \( \bar{\omega}_x \), respectively. Since the doubled time period is clearly seen, the Floquet mode belongs to mode C. Returning to Fig. 18, one can find modes C and I in the range of low \( G/d \leq 0.5 \). It should be noted that when the cylinder is positioned near the channel wall \( (G/d \leq 0.5) \), \( Re_{c2} \) is very close to

Fig. 14. Correlation coefficient at selected \( G/d \).

Fig. 15. Distribution of the mean pressure coefficient around the circular cylinder, \( Re = 150 \): (a) \( G/d = 0.75 \), (b) \( G/d = 1.75 \).

Fig. 16. Dominant Strouhal number (St).

Fig. 17. Variation of Floquet multiplier with spanwise wave number, \( G/d = 0.75 \).
meaning that 3D instability ensues almost simultaneously with the Hopf bifurcation. Mode I will be discussed in more detail in Section 3.7.

In the range of $G/d \geq 0.625$, mode QP is found near the critical condition. It is seen that unlike mode C, mode QP occurs at $Re_{c2}$ definitely different from $Re_{c1}$. Mode QP is locally less unstable (i.e. higher $Re_{c2}$) at $G/d = 1.75$. The sudden change in the trend of $Re_{c2}$ at $G/d = 1.0$ implies a topological change in the base flow between a "obstacle-near-wall regime" and a "vortex shedding regime", as shown in Fig. 6. It should be noted that only for $G/d = 0.625$, another critical mode other than modes C or QP is found at $Re_{c2} = 170.2, \lambda_c = 2.06$, which is indicated by the symbol $\nabla$. This mode has a converged Floquet multiplier and the same time period as that of the base flow, and does not show any spatial symmetry. The spanwise wavelength of the instability mode corresponding to the critical condition for each $G/d$ considered is presented in Fig. 20.

To elucidate evolution of Floquet modes, Fig. 21 shows variation of the growth rate of Floquet modes ($\sigma_I$) with $\beta$ near $Re_{c2}$ for the four selected values of $G/d$. The characteristics of the growth rate are heavily dependent upon $G/d$. In the case of $G/d = 0.25$, the sharp peak at $\beta \approx 1.2$ corresponding to mode C is dominant, whereas $\sigma_I$ is still slightly larger than 0 for very small $\beta$ even at $Re$ close to $Re_{c1}$, resulting in concurrent onset of the primary and the secondary instabilities (Fig. 21a). This trend is noticed in the range of $G/d \leq 0.5$. For $G/d = 0.625$, two definite peaks are identified at $\beta \approx 0.8$ and $\beta \approx 2.0$, corresponding to mode QP and a non-C non-QP mode ($\nabla$ in Figs. 18 and 20), respectively (Fig. 21b). In the range of $0.75 \leq G/d \leq 1.5$, the characteristics of $\sigma_I$ are similar to those for $G/d = 1.25$ shown in Fig. 21c. One single peak corresponding to mode QP is prevailing near $Re_{c2}$. For $G/d \geq 1.825$, a dominant peak is identified above $\beta \approx 3.0$ (see Fig. 20 also) corresponding to mode QP, as seen in Fig. 21d for $G/d = 1.75$. A local peak is noticed at $\beta \approx 0.8$, but the growth rate never exceeds zero.

3.7. Three-dimensional instability below $Re_{c1}$ ($G/d < 0.5$)

In this section, we study the 3D instability that occurs below $Re_{c1}$ (i.e. the base flow is steady). This instability is observed only in the range of $G/d \leq 0.5$. Fig. 22 shows time evolution of $N(t)$ for $\beta = 0.45$ and $0.7$ for $G/d = 0.25$. Here, the steady flow at $Re = 238$ was employed as the base flow. It should be noted that there is an unstable mode (here referred to as mode I as per Sheard et al. [17], $\beta = 0.45$) even though $Re = 238$ is below the critical Reynolds number of the primary instability ($Re_{c1} \approx 239.2$, Hopf bifurcation) obtained for 2D flow. Fig. 23 presents the growth rate of $N(t)$ plotted against $\beta$ for various values of $Re$. The critical Reynolds number for the 3D instability is identified as $Re_{c2} = 235.3$, and the dominant spanwise wave number is consistently found as $\beta = 0.44$ (corresponding to the spanwise wavelength, $\lambda_c = 2\pi/\beta = 14.27$). The mode I is different from modes C and QP in the sense that it appears when the 2D base flow is steady. Therefore, care must be taken in interpreting the stability analysis for Hopf bifurcation, especially in the cases of small $G/d$ where base-flow unsteadiness is suppressed by the cylinders periodically placed close to the channel wall. The critical Reynolds number of mode I is presented in Fig. 18. Contours of $\omega_0$ of the normalized Floquet mode for $G/d = 0.25, Re = 238, \beta = 0.45$ (mode I) are shown in Fig. 24, revealing no spatial symmetry. Mode I is also observed in baffled channel flow with a short baffle interval [40].

4. Conclusion

Instability characteristics in channel flow with a streamwise periodic array of circular cylinders have been numerically investigated. The flow configuration is a simple model for a heat exchanger with vortex generators to enhance heat transfer. Main emphasis was placed on understanding of the effects of the gap between the cylinder and the channel wall on the primary and the secondary instabilities. An immersed boundary method was employed to implement the cylinder within a Cartesian grid system.

It was found that flow topology significantly changes depending upon the gap, resulting in the related change of instability characteristics. When the cylinder is close to the channel wall, vortex shedding behind the cylinder is suppressed, while the gap flow induces flow separation on the channel wall leading to the recirculation regions travelling on the channel wall. However, when the cylinder is positioned away from the channel wall, vortex shedding in the downstream wake is active, and the flow separation on the channel wall disappears. The crossover of flow topology is reflected in the flow instabilities as well as the flow-induced forces on the
cylinder. The Stuart-Landau equation was utilized to compute the growth rate of the primary instability causing a Hopf bifurcation. The critical Reynolds number ($Re_{c1}$) was obtained for each $G/d$, revealing that the flow becomes more susceptible to the primary instability associated with the separated free shear layer behind the cylinder, rather than that associated with the flow separation on the channel wall.

The time-averaged drag coefficient increases with increasing $G/d$ in the entire range of $Re$ including both subcritical and supercritical regimes. It was shown that the shed vortices from the “upstream” cylinder affect the local streamwise velocity just upstream of the “downstream” cylinder, resulting in the $C_D$ increase. However, such an effect was not noticed for the recirculation regions travelling on the channel wall. The direction of lift force also depends on $G/d$. The lift force is upward for $G/d < 0.625$, while the opposite is the case for $G/d > 0.875$. This was explained by using pressure distribution on the cylinder for each case. The dominant $St$ of the 2D base flow turned out to increase as $G/d$ increases, but almost constant for each $G/d$.

A Floquet stability analysis was performed to elucidate the characteristics of the secondary instability leading to 3D flow.

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Fig. 21. Variation of the growth rate of Floquet modes with spanwise wave number: (a) $G/d = 0.25$, (b) $G/d = 0.625$, (c) $G/d = 1.25$, (d) $G/d = 1.75$.

Fig. 22. Temporal variation of the $L_2$ norm of the perturbation velocity field for $Re = 238$ and $G/d = 0.25$.

Fig. 23. Variation of the growth rate of Floquet modes with spanwise wave number for $G/d = 0.25$.

Fig. 24. Contours of the streamwise vorticity component ($\omega_x$) of the normalized Floquet mode (mode I), $G/d = 0.25$, $Re = 238$, $\beta = 0.45$. 
The critical Reynolds number ($Re_c$) and the associated spanwise wavelength were presented for each $G/d$ considered here. Three distinct modes were identified, namely, mode C, mode QP, and mode I. Their occurrence and the associated dominant spanwise wave numbers strongly depend upon $G/d$. In particular, when the cylinder is close to the channel wall ($G/d \leq 0.5$), a 3D instability (mode I) occurs even below $Re_c$. Criticality of a non-C, non-QP mode was also found exclusively at $G/d = 0.625$. The current findings shed light on complete understanding of the flow-instability characteristics in the heat exchangers with vortex generators to enhance heat transfer.

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References