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Dynamics and guided waves in a smart Timoshenko beam with lateral contraction

I Park, S Kim and U Lee

Department of Mechanical Engineering, Inha University, 253 Yonghyun-dong, Nam-gu, Incheon 402-751, Republic of Korea

E-mail: ulee@inha.ac.kr

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Abstract
Surface-bonded wafer-type piezoelectric transducers (PZTs) have been widely used to excite or measure ultrasonic guided waves for the structural health monitoring of thin-walled structures. For successful prediction of the dynamics and ultrasonic guided waves, it is essential to use very reliable computational models for the PZT-bonded multi-layer smart structures. In this paper, the spectral element model is developed for two-layer smart beams which consist of a metallic base beam layer and a PZT layer. Axial-bending-shear-contraction coupled equations of motion and boundary conditions are derived by using Hamilton’s principle with Lagrange multipliers based on the Timoshenko beam theory and Mindlin–Herrmann rod theory. The high accuracy of this spectral element model is verified in due course and the effects of a lateral contraction on the dynamics and guided wave characteristics of the example smart beams are investigated by using this spectral element model. In addition, the constraint forces at the interface between the base beam and the PZT layer are also investigated via Lagrange multipliers.

1. Introduction
Ultrasonic guided waves have recently received a great deal of attention as a promising means to inspect and monitor the health of thin-walled structures. In order to generate ultrasonic guided waves in a structure, wafer-type piezoelectric transducers made of lead zirconate titanate (PZTs) have been widely used [1]. As the PZTs are usually bonded to the surfaces of a structure, the PZT-bonded structure becomes a multi-layer structure [1]. As there can be numerous types of multi-layer structure, the discussion in this paper will be confined to two-layer smart beams which consist of a metallic base beam layer (simply, base beam) and a PZT layer.

Dynamic coupling between the base beam and the PZT layer must be modeled in an accurate way for successful prediction of the dynamics and ultrasonic guided waves generated in the smart beam. There have been numerous analytical models for various multi-layer structures in the literature. Elastic–elastic two-layer beams [2], elastic–viscoelastic two-layer beams [3], and elastic–piezoelectric two-layer smart beams [4–6] are typical examples of two-layer beams. Despite numerous analytical models for multi-layer beams, the finite element method (FEM) has been widely adopted because it is certainly a powerful computational method to deal with complex engineering problems [7, 8]. However, as a drawback of FEM, very fine meshing is often required to improve the solution accuracy, especially at high frequency, which results in a significant increase of computation cost. The spectral element method (SEM) can be considered as an alternative to FEM because it can provide extremely accurate solutions even at very high frequency by using the spectral element matrix (often called the exact dynamic stiffness matrix) formulated from frequency-dependent (dynamic) shape functions.

From the historical point of view, the fundamental concept of SEM was first introduced by Narayanan and Beskos [9] and later named ‘SEM’ by Doyle [10]. The SEM has been successfully applied to wave propagation and
dynamics of structures [10, 11]. The standard FEM is a time domain method because it is normally formulated in the time \((t)\) domain in a well-known matrix form of ordinary differential equations, \(\ddot{M}d(t) + \dot{C}d(t) + \ddot{K}d(t) = f(t)\), where \(M\), \(C\), and \(K\) are the finite element mass matrix, damping matrix, and stiffness matrix, respectively, \(d\) is the nodal degrees of freedom (DOFs) vector, and \(f\) is the nodal forces vector. On the other hand, SEM is a frequency domain method because it is formulated in the frequency \((\omega)\) domain in a matrix form of algebraic equations, \(S(\omega)d(\omega) = f(\omega)\), where \(S(\omega)\) is the frequency-dependent spectral nodal DOFs matrix, \(d(\omega)\) is the spectral nodal DOFs vector, and \(f(\omega)\) is the spectral nodal forces vector. For the readers, it is worth clarifying that the \(p\)-version FEM by Patera [12], which is unfortunately given the same name ‘SEM’ in the literature, is a time domain method which is completely different from the present ‘frequency domain’ SEM. In the present SEM, the spectral element matrix is formulated in the frequency domain by using the frequency-dependent shape functions derived from exact free wave solutions. Thus, in theory, SEM will provide exact frequency domain solutions by using the minimum number of DOFs [11]. Accurate time domain solutions (time histories) can be readily obtained from the frequency domain solutions by using the FFT-based spectral analysis method.

Despite various advantages of SEM, there have been very few applications to multi-layer smart structures. Lee and Kim [6] were the first to develop a spectral element model for two-layer smart Bernoulli–Euler beams which consisted of a metallic base beam layer and a PZT layer. Later on, they also developed spectral element models for three-layer smart beams which consisted of a metallic base beam layer, a viscoelastic layer, and a PZT layer [13] and for smart composite beams which consisted of a composite laminated base beam layer and a PZT layer [14]. Recently Park et al [15] developed a spectral element model for two-layer smart Timoshenko beams which consisted of a metallic base beam layer and a PZT layer. To formulate the spectral element model, they represented displacement fields in terms of ten wave modes by using ten wavenumbers computed from the dispersion relation. It is natural to define as many nodal DOFs as the number of wave modes in order to properly express wave mode magnitudes in terms of the nodal DOFs. However, they defined only four nodal DOFs at each node (missing one nodal DOF for the rotation of the PZT layer) to result in an ill-posed problem. Thus, they had to utilize a reduction process to derive an eight-by-eight spectral element matrix with eight spectral nodal DOFs.

The purposes of this paper are (1) to formulate a spectral element model for two-layer smart beams which consist of a metallic base beam layer and a PZT layer. In this formulation, ten spectral nodal DOFs are defined to be consistent with ten wave modes which are superposed to represent the displacement fields in a smart beam; (2) to evaluate the accuracy of this spectral element model by comparing with numerical results obtained by using the commercial FE analysis package ANSYS and the experimental results; and (3) to investigate the effects of the lateral contraction and PZT layer on the dynamics and guided waves characteristics of example smart beams.

2. Derivation of the equations of motion

Consider a uniform two-layer smart beam which consists of a metallic base beam layer (simply, base beam) of thickness \(h_b\) and a PZT layer of thickness \(h_p\). The smart beam has the width \(b\), length \(L\), and total thickness \(h = h_b + h_p\) and is subjected to plane deformation in the \((x, z)\)-plane, as shown in figure 1. The global coordinate system \((x, y, z)\) is located on the mid-plane of the base beam so that the \(x\)-axis coincides with the beam axis.

By applying the Timoshenko beam theory [16] and Mindlin–Herrmann rod theory [17] to the base beam, the effects of the shear deformation, rotary inertia, and lateral contraction in the thickness \(z\)-direction can be all taken into account in the displacement fields as

\[
\begin{align*}
\delta(x, z, t) &= \delta_0(x, t) + z\theta_b(x, t) \\
\psi_b(x, z, t) &= \psi_0(x, t) + z\psi_b(x, t)
\end{align*}
\]

(1)

where \(\delta_b(x, t)\) is the axial displacement, \(\psi_b(x, t)\) is the transverse displacement, \(\theta_b(x, t)\) is the rotation of the cross-section normal to the mid-plane of the base beam about the \(y\)-axis, and \(\psi_b(x, t)\) is the lateral contraction in the thickness direction. The displacement fields in the thin PZT layer are assumed as

\[
\begin{align*}
\delta_p(x, \bar{z}, t) &= \bar{\delta}_0(x, t) - \bar{z}\bar{\psi}_p(x, t) \\
\psi_p(x, \bar{z}, t) &= \bar{w}_p(x, t)
\end{align*}
\]

(2)
where the prime (') denotes the derivative with respect to $x$ and $\bar{z}$ is measured with respect to the mid-plane of the PZT layer, as shown in figure 1(b). Assuming that the PZT layer is perfectly bonded to the upper surface of the base beam, two constraint equations can be obtained from the connectively conditions as follows:

$$g_1(x, t) = u_0(x, t) - u_0(x, t) + \frac{1}{2} h_0 \theta_0(x, t) = 0$$

$$g_2(x, t) = w_0(x, t) - w_0(x, t) + \frac{1}{2} h_0 \psi_0(x, t) = 0.$$  (3)

The equations of motion and the associated boundary conditions for two-layer smart beams can be derived from Hamilton’s principle with Lagrange multipliers given by [18]

$$\int_{t_1}^{t_2} \left\{ \delta T - \delta V + \delta W + \delta \int_0^L (\lambda_1 g_1 + \lambda_2 g_2) \, dx \right\} \, dt = 0  \quad (4)$$

where $V$ and $T$ are strain and kinetic energies, respectively, and $\delta W$ is the virtual work done by external loadings as well as by the voltage applied to the PZT layer. The operator $\delta$ is the variational symbol. $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers which are related to shearing (tangential) and peeling (normal) constraint forces at the interface between the base beam and the PZT layer, respectively.

By using the constitutive relations for the metallic base beam and the PZT layer provided in appendices A and B, the strain and kinetic energies can be obtained as

$$V = \frac{1}{2} \int_0^L \left[ C_{111} A_b u_0'^2 + 2 C_{133} A_b \psi_0 u_0' + E_b \theta_0'^2 + \kappa G A_b (w_0'^2 - \theta_0'^2) + \kappa G L \psi_0'^2 + C_{11} A_b \psi_0'^2 + \bar{c}_{111} A_b u_0'^2 + \bar{c}_{111} A_b w_0'^2 + \beta_3 A_b D_3^2 \right] \, dx$$

$$T = \frac{1}{2} \int_0^L \left[ \rho_0 h_0 \ddot{w}_0 + \rho_0 h_0 \ddot{\psi}_0 + \rho_0 A_b \ddot{\theta}_0 + \rho_0 h_0 \ddot{\theta}_0 + \frac{1}{2} \rho_0 h_0 \ddot{\psi}_0 \right] \, dx + \int_0^L b V(t) \delta \theta_0 \, dx$$  (5)

where $C_{111}, C_{133}$ and $C_{33}$ are elastic properties of the base beam defined in appendix A, $\bar{c}_{111}$ and $\beta_3$ and $h_3$ are material properties of the PZT layer defined in appendix B, $\rho_b$ and $\rho_0$ are the mass densities of the base beam and PZT layer, respectively, and $A_b = b h_0$, $A_p = b h_p$, $h_b = \frac{1}{12} b h_0^3$, and $I_p = \frac{1}{12} b h_p^3$. Finally, the virtual work $\delta W$ can be obtained as

$$\delta W = N_1 \delta \theta_0(0) + N_2 \delta \theta_0(L) + M_1 \delta \theta_0(0) + M_2 \delta \theta_0(L) + Q_1 \delta w_0(0) + Q_2 \delta w_0(L) + \int_0^L [f_u(x, t) \delta \theta_0 + f_w(x, t) \delta w_0 + f_0(x, t) \delta \theta_0] \, dx + \int_0^L b V(t) \delta \theta_0 \, dx$$  (6)

where $N_i, M_i$ and $Q_i$ $(i = 1, 2)$ are the resultant axial forces, bending moments and transverse shear forces applied at the boundaries $x = 0$ and $L$, $f_u(x, t)$, $f_w(x, t)$ and $f_0(x, t)$ are distributed loadings and $V(t)$ is the voltage applied to the PZT layer.

By substituting equations (3), (5) and (6) into (4) and then taking the integral by parts, we can obtain equations of motion, boundary conditions, and complementary equations for the electrical displacement $D_3$ and Lagrange multipliers $\lambda_1$ and $\lambda_2$ as follows.

(a) Equations of motion

$$EA_w u_0'''' - K_2 w_0'''' - K_1 \theta_0'' + K_3 \psi_0'' = -m_u \ddot{w}_0 + m_u \theta_0 + f_u = 0$$

$$-EI_p w_0'''' + \kappa G A_b w_0'' + K_2 \psi_0'' - K_4 \theta_0'' - K_5 \psi_0'' - K_6 \psi_0'' - m_w \ddot{w}_0 - m_w \psi_0 + f_w = 0$$

$$E \theta_0'' - \kappa G A_b \theta_0 + K_3 \psi_0'' + K_4 \psi_0'' + K_5 \psi_0'' + K_7 \psi_0'' \quad (7)$$

$$-m_\theta \ddot{\theta}_0 + m_\theta \ddot{w}_0 + f_\theta = 0$$

$$-E \psi_0'' + \kappa G I \psi_0'' - EA \psi_0 - K_3 u_0'' + K_4 u_0'' = -K_6 \psi_0'' + K_7 \psi_0'' - m_\psi \ddot{\psi}_0 - m_\psi \ddot{w}_0 = 0$$

where the effective structural and inertia properties of the smart beam are defined in appendix C.

(b) Boundary conditions

$$N_b(0, t) + N_p(0, t) = -N_1 \quad \text{or} \quad \delta \theta_0(0, t) = 0$$

$$N_b(L, t) + N_p(L, t) = N_2 \quad \text{or} \quad \delta \theta_0(L, t) = 0$$

$$Q_b(0, t) - M_p'(0, t) + \frac{1}{2} h_0 \psi_0(0, t) = 0 \quad \text{or} \quad \delta w_0(0, t) = 0$$

$$Q_b(L, t) - M_p'(L, t) + \frac{1}{2} h_0 \psi_0(L, t) = 0 \quad \text{or} \quad \delta w_0(L, t) = 0$$

$$M_b(0, t) - \frac{1}{2} h_0 N_p(0, t) = -M_1 \quad \text{or} \quad \delta \theta_0(0, t) = 0$$

$$M_b(L, t) - \frac{1}{2} h_0 N_p(L, t) = M_2 \quad \text{or} \quad \delta \theta_0(L, t) = 0$$

$$R_b(0, t) - \frac{1}{2} h_0 M_p'(0, t) + \frac{1}{2} h_0 h_p N_p'(0, t) = 0 \quad \text{or} \quad \delta \psi_0(0, t) = 0$$

$$R_b(L, t) - \frac{1}{2} h_0 M_p'(L, t) + \frac{1}{2} h_0 h_p N_p'(L, t) = 0 \quad \text{or} \quad \delta \psi_0(L, t) = 0$$

$$M_p(0, t) - \frac{1}{2} h_0 N_p(0, t) = 0 \quad \text{or} \quad \delta \psi_0(0, t) = 0$$

$$M_p(L, t) - \frac{1}{2} h_0 N_p(L, t) = 0 \quad \text{or} \quad \delta \psi_0(L, t) = 0$$

where $N_b(x, t), Q_b(x, t), M_b(x, t), R_b(x, t), N_p(x, t), M_p(x, t)$ are the resultant forces and moments defined by

$$N_b(x, t) = C_{111} A_b \ddot{w}_0 + C_{133} A_b \ddot{\psi}_0$$

$$Q_b(x, t) = \kappa G A_b (w_0'' - \theta_0'')$$

$$M_b(x, t) = E \ddot{\theta}_0$$

$$R_b(x, t) = \kappa G \ddot{\psi}_0$$

$$N_p(x, t) = E A_p [w_0'' - \frac{1}{2} h_0 \psi_0''] - \frac{1}{2} h_0 (w_0'' + \frac{1}{2} h_0 \psi_0'')$$

$$M_p(x, t) = \ddot{c}_{111} A_p (w_0'' + \frac{1}{2} h_0 \psi_0'')$$  (9)

(c) Complementary equations

$$D_3 = \bar{h}_{31} \bar{p}_3^{1-1} w_0'' + b A_p \bar{p}_3^{1-1} V(t)$$

$$= \bar{h}_{31} \bar{p}_3^{1-1} [u_0'' - \frac{1}{2} h_0 \psi_0''] + \frac{1}{2} h_0 \bar{h}_{31} \bar{p}_3^{1-1} V(t) \times (w_0'' + \frac{1}{2} h_0 \psi_0'') + b A_p \bar{p}_3^{1-1} V(t)$$  (10)
and
\[
\lambda_1 = -EA_p\left[u'''_{b0} - \frac{1}{2}b_0\theta''_{b0} - \frac{1}{2}h_p\left(v''_{b0} + \frac{1}{2}\psi''_{b0}\right)\right]
\]
\[
\lambda_2 = -\frac{1}{2}h_pEA_p\left[u'''_{b0} - \frac{1}{2}h_0\theta''_{b0}\right] + EI_p\left[w'''_{b0} + \frac{1}{2}h_0\psi''_{b0}\right].
\]

If all the terms associated with the PZT layer are removed from equation (7), the second and third equations are completely decoupled from the first and fourth equations so that they are reduced to the equations of motion for the standard Timoshenko beam model [16] while the first and fourth equations are reduced to those for the standard Mindlin–Herrmann rod model [17]. Although the present equations of motion (7) have the same forms as those presented by Park et al [15], they are different from each other because the present equations of motion are derived by applying the plane stress-induced constitutive relations to both axial vibration and transverse bending vibration of the smart beam, while the equations of motion by Park et al [15] are derived by applying the plane strain-induced constitutive relations to the axial vibration and the plane stress-induced constitutive relations to the transverse bending vibration. The present equations of motion become identical to those by Park et al [15] when \( C_{11} = C_{33} = E/(1 - \nu^2) \), \( C_{13} = \nu E/(1 - \nu^2) \), \( \rho_2^{35} \) and \( h_{31} \) are replaced with \( \lambda + 2\mu, \lambda, \rho_2^{35}, \) and \( h_{31} \), respectively. In this study, it is numerically shown that the present equations of motion provide solutions that are much closer to the results obtained by using the commercial FE analysis package ANSYS, when compared with the equations of motion by Park et al [15].

3. Formulation of the spectral element model

3.1. Governing equations in the frequency domain

As the first step of the spectral element formulation, the equations of motion need to be transformed into the frequency domain by using the discrete Fourier transform (DFT) theory [19]. Thus, all dependent variables and external forces are represented in the spectral form as
\[
[u_0(x, t), w_0(x, t), \theta_0(x, t), \psi_0(x, t)] = \frac{1}{N} \sum_{n=0}^{N-1} [u_n(x), w_n(x), \Theta_n(x), \Psi_n(x)] e^{i\omega_n t}
\]
and
\[
[f_u(x, t), f_w(x, t), f_\theta(x, t), f_\psi(x, t)] = \frac{1}{N} \sum_{n=0}^{N-1} [F_u_n(x), F_w_n(x), F_\theta_n(x), F_\psi_n(x)] e^{i\omega_n t}
\]
where \( i = \sqrt{-1} \) is the imaginary unit, \( W_n = [u_n, w_n, \Theta_n, \Psi_n]^T \) and \( P_n = [F_u_n, F_w_n, F_\theta_n]^T \) are the spectral (Fourier) components of the dependent variables and external forces, respectively, and \( N \) is the number of spectral components to be considered in the FFT-based spectral analysis. Substituting equations (12) and (13) into (7) yields the governing equations in the frequency domain as
\[

EA_p U'''' - K_2 W'''' - K_1 \Theta'''' + K_3 \Psi'' + K_4 \Psi'''' + m_{uu}\omega^2 U
\]
\[
\quad - m_{uw}\omega^2 \Theta + F_u = 0
\]
\[
\quad -EI_p W'''' + \kappa GA_b W'' + K_2 U'''' - K_4 \Theta'''' - K_5 \Theta''
\]
\[
\quad - K_6 \Psi'''' + m_{ww}\omega^2 W + m_{u\psi}\omega^2 \Psi + F_w = 0
\]
\[
EI\Theta'''' - \kappa GA_b \Theta - K_1 U'''' + K_4 \Psi'' + K_5 W' + K_7 \Psi''
\]
\[
\quad + m_{u\phi}\omega^2 \Theta - m_{w\phi}\omega^2 U + F_\theta = 0
\]
\[
-EJ\Psi'''' + \kappa GI\Psi'' - EA_\phi \Psi - K_3 U'' + K_4 \Psi''
\]
\[
\quad - K_6 W'''' - K_7 \Theta'''' + m_{\phi\psi}\omega^2 \Psi + m_{\phi\phi}\omega^2 W = 0
\]

where the subscripts \( n \) are omitted for the sake of brevity. Similarly, by representing all resultant forces and moments, boundary forces and moments, and applied voltage in the spectral forms, the natural boundary conditions (8) and the force–displacement relations (9) can be transformed into the frequency domain.

3.2. Free waves and dispersion relation

The homogeneous governing equations can be reduced from equation (14) by removing all external forces as
\[

EA_p U'''' - K_2 W'''' - K_1 \Theta'''' + K_3 \Psi'' + K_4 \Psi''''
\]
\[
\quad + m_{uu}\omega^2 U - m_{uw}\omega^2 \Theta = 0
\]
\[
\quad -EI_p W'''' + \kappa GA_b W'' + K_2 U'''' - K_4 \Theta'''' - K_5 \Theta''
\]
\[
\quad - K_6 \Psi'''' + m_{ww}\omega^2 W + m_{u\psi}\omega^2 \Psi = 0
\]
\[
EI\Theta'''' - \kappa GA_b \Theta - K_1 U'''' + K_4 \Psi'' + K_5 W' + K_7 \Psi''
\]
\[
\quad + m_{u\phi}\omega^2 \Theta - m_{w\phi}\omega^2 U = 0
\]
\[
-EJ\Psi'''' + \kappa GI\Psi'' - EA_\phi \Psi - K_3 U'' + K_4 \Psi''
\]
\[
\quad - K_6 W'''' - K_7 \Theta'''' + m_{\phi\psi}\omega^2 \Psi + m_{\phi\phi}\omega^2 W = 0
\]

The solutions of equation (15) can be assumed as
\[
\{U(x) W(x) \Theta(x) \Psi(x)\} = [1 \ r_w \ r_\theta \ r_\psi] a e^{-ikx}
\]
where \( a \) is a constant and \( k \) represents the wavenumber. Substituting equation (16) into (15) yields an eigenvalue problem as
\[
\begin{bmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & 1 & 0 \\
X_{21} & X_{22} & X_{23} & X_{24} & r_w & 0 \\
X_{31} & X_{32} & X_{33} & X_{34} & 0 & r_\theta \\
X_{41} & X_{42} & X_{43} & X_{44} & 0 & r_\psi
\end{bmatrix}
\begin{bmatrix}
a \\
r_w \\
r_\theta \\
r_\psi
\end{bmatrix}
= 0
\]

where the matrix components \( X_{ij} \) are defined in appendix D.

The dispersion equation can be derived from equation (17) in the form of
\[
\chi^5 + c_1 \chi^4 + c_2 \chi^3 + c_3 \chi^2 + c_4 \chi + c_5 = 0
\]
(\( \chi = k^2 \)).

From equation (18), ten wavenumbers can be obtained as
\[
k_m = -k_{m+5} = \sqrt{\chi_m} \quad (m = 1, 2, \ldots, 5)
\]
For each wavenumber \( k \) \( (j = 1, 2, \ldots, 10) \), the coefficients \( r_w \), \( r_\theta \), and \( r_\psi \) can be determined from equation (17) as

\[
\begin{bmatrix}
  r_w \\
  r_\theta \\
  r_\psi_{j}
\end{bmatrix} = - \begin{bmatrix}
  X_{22} & X_{23} & X_{24} \\
  X_{32} & X_{33} & X_{34} \\
  X_{42} & X_{43} & X_{44}
\end{bmatrix}^{-1} \begin{bmatrix}
  X_{21} \\
  X_{31} \\
  X_{41}
\end{bmatrix}
\]

for \( k = k_j \) \( (j = 1, 2, 3, \ldots, 10) \). (20)

By using equation (20), three ten-by-ten diagonal matrices can be defined as follows:

\[
R_w = \text{diag} \left[ r_{wj} \right] \quad R_\theta = \text{diag} \left[ r_{\theta j} \right] \\
R_\psi = \text{diag} \left[ r_{\psi j} \right] \quad (j = 1, 2, 3, \ldots, 10).
\]

(21)

By using equations (20) and (21), the general solutions of equation (15) can be expressed as

\[
U(x; \omega) = \sum_{j=1}^{10} a_j e^{-i k_j x} = e(x; \omega)a
\]

\[
W(x; \omega) = \sum_{j=1}^{10} r_{wj} a_j e^{-i k_j x} = e(x; \omega)R_w a
\]

\[
\Theta(x; \omega) = \sum_{j=1}^{10} r_{\theta j} a_j e^{-i k_j x} = e(x; \omega)R_\theta a
\]

\[
\Psi(x; \omega) = \sum_{j=1}^{10} r_{\psi j} a_j e^{-i k_j x} = e(x; \omega)R_\psi a
\]

where

\[
e(x; \omega) = [e^{-i k_1 x} \ e^{-i k_2 x} \ e^{-i k_3 x} \ \ldots \ e^{-i k_{10} x}]
\]

and

\[
a = [a_1 \ a_2 \ a_3 \ \ldots \ a_{10}]^T.
\]

Physically, equation (22) represents the free waves that are propagating in the two-layer smart beam.

### 3.3. Spectral nodal DOFs and dynamic shape functions

Consider a finite smart beam element of length \( l \). The spectral components of the nodal degrees of freedom (DOFs) (simply, spectral nodal DOFs) are defined as follows:

\[
\begin{align*}
U_1 &= U(0) & U_2 &= U(l) \\
W_1 &= W(0) & W_2 &= W(l) \\
\Theta_{\theta 1} &= \Theta(0) & \Theta_{\theta 2} &= \Theta(l) \\
\Psi_1 &= \Psi(0) & \Psi_2 &= \Psi(l) \\
\Theta_{p 1} &= W'(0) + \frac{1}{2} b_h \Psi'(0) & \Theta_{p 2} &= W'(l) + \frac{1}{2} b_h \Psi'(l).
\end{align*}
\]

(25)

Applying equation (22) to equation (25) gives a relationship as

\[
d = H(\omega)a
\]

(26)

where \( d \) is the spectral nodal DOFs vector defined by

\[
d = [U_1 W_1 \Theta_{\theta 1} \Psi_1 \Theta_{p 1} \ U_2 W_2 \Theta_{\theta 2} \Psi_2 \Theta_{p 2}]^T
\]

and \( H(\omega) \) is the ten-by-ten matrix defined in appendix E.

The constant vector \( a \) can be eliminated from equation (22) by using equation (26) to rewrite general solutions in terms of \( d \) as follows:

\[
\begin{align*}
U(x; \omega) &= N_u(x; \omega) d \\
W(x; \omega) &= N_w(x; \omega) d \\
\Theta(x; \omega) &= N_\theta(x; \omega) d \\
\Psi(x; \omega) &= N_\psi(x; \omega) d
\end{align*}
\]

(28)

where \( N_u(x; \omega) \), \( N_w(x; \omega) \), \( N_\theta(x; \omega) \), and \( N_\psi(x; \omega) \) are frequency-dependent shape functions defined by

\[
\begin{align*}
N_u(x; \omega) &= e(x; \omega) H^{-1}_u \\
N_w(x; \omega) &= e(x; \omega) R_w H^{-1}_w \\
N_\theta(x; \omega) &= e(x; \omega) R_\theta H^{-1}_\theta \\
N_\psi(x; \omega) &= e(x; \omega) R_\psi H^{-1}_\psi
\end{align*}
\]

(29)

### 3.4. Formulation of the spectral element equation

The variational method [11] can be used to formulate the spectral element model for a finite smart beam element. To that end, the weak form of equation (14) is obtained from the weighted-integral statement after taking the integral by parts and applying natural boundary conditions as follows:

\[
\int_0^l \left[ E_a U' \delta U' + E_I \Theta' \delta \Theta' + E_A \Psi' \delta \Psi' + E_L \psi' \delta \psi' \\
+ \kappa G_{ab}(\Theta \delta \Theta + W' \delta W') + \kappa G_{1} \Psi' \delta \Psi' + E_{I} \psi' \delta \psi' \\
- K_{1}(\Theta \delta U' + U' \delta \Theta') - K_{2}(W' \delta U' + U' \delta W') \\
+ K_{3}(\Psi \delta U' + U' \delta \Psi') + K_{4}(\Theta \delta W' + W' \delta \Theta') \\
- U' \delta \Psi' - \Psi' \delta U' - K_{5}(\Theta \delta W' + W' \delta \Theta) \\
+ K_{6}(\Psi' \delta W' + W' \delta \Psi') + K_{7}(\Theta' \delta \Theta' + \Theta \delta \Theta') \\
+ \omega^2 \left(-m_{uu} U \delta U + m_{ww} W \delta W + (m_{u w}) \delta \Theta U \\
+ m_{w w} \delta W - m_{u w} \Psi \delta W - m_{p w} \Psi \delta W + (m_{p p} \delta \Theta + m_{p \psi} \Psi \delta \Theta + m_{\phi \psi} \Psi \delta \Theta)ight)\right] \ dx = \delta d^T f_m^{\text{mech}} + \delta d^T f^{\text{piezo}}
\]

(30)

where

\[
f_m^{\text{mech}}(\omega) = \begin{bmatrix}
\tilde{N}_1 \tilde{Q}_1 M_1 0 0 \tilde{N}_2 \tilde{Q}_2 M_2 0 0
\end{bmatrix}^T
\]

(31)

\[
f^{\text{piezo}}(\omega) = \beta_{31} \tilde{S}_{11} \tilde{S}_{33}^{-1} \tilde{V}(\omega)
\]

(32)

In equations (31) and (32), the quantities with over-bars denote the spectral components of the corresponding time domain quantities.

Finally, substituting equation (28) into (30) gives the spectral element equation in the form

\[
S_A(\omega)d = f_A(\omega)
\]

(33)

where

\[
S_A(\omega) = H(\omega)^{-T} D(\omega) H(\omega)^{-1}
\]

(34)
\[ f_A(\omega) = \int_0^l \left[ [\mathbf{N}^T_f F_u + \mathbf{N}^T_w F_w + \mathbf{N}^T_p F_p] \right] \, dx + \mathbf{f}_{\text{mech}}^\epsilon(\omega) + \mathbf{f}_{\text{piezo}}(\omega) \]  \hspace{1cm} (35)

and the matrix \( \mathbf{D}(\omega) \) is defined in appendix F. \( \mathbf{S}_A(\omega) \) is the ten-by-ten symmetric spectral element matrix (often called the exact dynamic stiffness matrix) and \( f_A(\omega) \) is the spectral nodal forces vector. So far, we have derived the spectral element equation when the surface-bonded PZT is used as an actuator.

Now consider the case when the surface-bonded PZT is used as a sensor. In this case, the following open-circuit condition must be satisfied in addition [15, 20]:

\[ \int_0^l b h_p D_3 \, dx = 0. \] \hspace{1cm} (36)

Substituting equation (10) (after transforming into the frequency domain) into equation (36) gives

\[ \ddot{V}(\omega) = -\ddot{h}_{31} h_p \gamma^{-1} \times [1 \quad 0 \quad \frac{1}{2}h_0 \quad 0 \quad \frac{1}{2}h_p \quad 0 \quad \frac{1}{2}h_0 \quad 0 \quad \frac{1}{2}h_p] \mathbf{d}. \] \hspace{1cm} (37)

Substituting equation (37) into (32) gives

\[ \mathbf{f}_{\text{piezo}}(\omega) = -\ddot{\mathbf{S}} \mathbf{d} \] \hspace{1cm} (38)

where \( \ddot{\mathbf{S}} \) is the ten-by-ten matrix defined in appendix G.

Applying equation (38) to equation (35) and then the result to equation (33) gives a new form of spectral element equation as

\[ \mathbf{S}_\delta(\omega) \mathbf{d} = \mathbf{f}_\delta(\omega) \] \hspace{1cm} (39)

where

\[ \mathbf{S}_\delta(\omega) = \mathbf{S}_A(\omega) + \ddot{\mathbf{S}} \] \hspace{1cm} (40)

\[ f_\delta(\omega) = \int_0^l \left[ [\mathbf{N}^T_f F_u + \mathbf{N}^T_w F_w + \mathbf{N}^T_p F_p] \right] \, dx + \mathbf{f}_{\text{mech}}^\epsilon(\omega). \] \hspace{1cm} (41)

Equation (39) is the spectral element equation when the surface-bonded PZT is used as a sensor.

The applicability of the present spectral element model will be limited to the frequency regime where the governing equations of motion (7) derived based on the Timoshenko beam theory and the Mindlin–Herrmann rod theory are valid. In the higher frequency regime, more elaborate higher order beam theories need to be adopted.

4. Numerical examples and discussion

4.1. Evaluation of the present spectral element model

Figure 2 shows an example cantilevered smart beam to be considered for the numerical evaluation of the present spectral element model. The material properties of the metallic base beam and PZT layer are given in tables 1 and 2, respectively. As exact analytical or numerical solutions are not available in the literature for the typical example smart beam, the accuracy of the present spectral element model is evaluated by comparing with the results obtained by the commercial FE analysis package ANSYS [21] and experiments.

![Figure 2. An example cantilevered smart beam partially covered with a PZT layer (units: mm).](image-url)

![Figure 3. Frequency response functions of the axial displacement obtained by the present SEM(n) and ANSYS(n), where n = total number of finite elements used in the analysis.](image-url)

<table>
<thead>
<tr>
<th>Table 1. Material properties of the metallic base beam.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material: 6061-T6 aluminum</td>
</tr>
<tr>
<td>( E = 69 \text{ GPa} )</td>
</tr>
<tr>
<td>( v = 0.33 )</td>
</tr>
<tr>
<td>( \rho_0 = 2700 \text{ kg m}^{-3} )</td>
</tr>
</tbody>
</table>

The frequency response functions (FRFs) obtained by the present spectral element model (denoted by SEM) and ANSYS are compared in figure 3 for the axial displacement and in figure 4 for the transverse displacement. Hysteretic material damping is taken into account for the metallic base beam by utilizing a complex modulus \( E^* = E(1+i\eta) \) with \( \eta = 0.005 \) [15, 27]. Only three finite elements (two regular metallic beam elements and one smart beam element) suffice to obtain the SEM results. For the ANSYS results, a 2D FE analysis is conducted using the eight-node quadratic plane stress elements. The number of finite elements is increased until sufficiently converged results are obtained. In figures 3 and 4, two ANSYS results are displayed: one is obtained by using a total of 400 finite elements (300 elements for the base beam layer (size: 2 mm × 2 mm/each element); 100 elements for the PZT layer (size: 2 mm × 0.5 mm/each element).
Table 2. Material properties of the PZT layer.

<table>
<thead>
<tr>
<th>Material: PZT-5A</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{11} = 124.26 \text{ GPa} )</td>
</tr>
<tr>
<td>( c_{12} = 79.092 \text{ GPa} )</td>
</tr>
<tr>
<td>( c_{13} = 63.550 \text{ GPa} )</td>
</tr>
<tr>
<td>( c_{22} = 124.26 \text{ GPa} )</td>
</tr>
<tr>
<td>( c_{23} = 63.550 \text{ GPa} )</td>
</tr>
<tr>
<td>( c_{33} = 144.91 \text{ GPa} )</td>
</tr>
<tr>
<td>( k_{31} = -7.3116 \times 10^8 \text{ V m}^{-1} )</td>
</tr>
<tr>
<td>( k_{32} = -7.3116 \times 10^8 \text{ V m}^{-1} )</td>
</tr>
<tr>
<td>( k_{33} = 21.566 \times 10^8 \text{ V m}^{-1} )</td>
</tr>
<tr>
<td>( \rho_s = 1.3664 \times 10^8 \text{ m F}^{-1} )</td>
</tr>
<tr>
<td>( \rho_p = 7750 \text{ kg m}^{-3} )</td>
</tr>
</tbody>
</table>

Figure 4. Frequency response functions of the transverse displacement obtained by the present SEM (n) and ANSYS(n), where n = total number of finite elements used in the analysis.

Figure 5. An example cantilevered beam considered for the simulation of guided waves (units: mm).

Figure 6. Guided waves predicted by the present SEM(n) and ANSYS(n), where n = total number of finite elements used in the analysis.

sensor rather than as an actuator; this can be readily expected through equation (40).

The guided waves predicted by the present SEM and ANSYS are also compared. As shown in figure 5, the guided waves are excited in a cantilevered uniform metallic beam of length 2100 mm by using a PZT actuator at 600 mm distance from the fixed end and then measured by a PZT sensor at 600 mm distance from the free end of the beam. The input voltage applied to the PZT actuator has the form of a Morlet wavelet signal (100 kHz center frequency, 100 peak-to-peak voltages with five cycles). For the SEM results displayed in figure 6, only five finite elements (three regular metallic beam elements and two smart beam elements) are used. The first (small) wavepackets in figure 6 represent the symmetric longitudinal waves (denoted by \( S_0 \) mode) and the second (large) wavepackets represent the anti-symmetric transverse waves (denoted by \( A_0 \) mode). The guided waves predicted by ANSYS using 16 820 finite elements are found to be much closer to the SEM results when compared with those predicted by using 4210 finite elements; this observation also verifies the high accuracy of the present spectral element model. It is worth mentioning that the CPU times required to compute the guided waves shown in figure 6 are about 58 s for SEM(5), 4823 s for ANSYS(4210), and 30 985 s for ANSYS(16820).

The present SEM model is also evaluated by comparing with experimental results. Figure 7 represents the comparison of the flexural wave envelope predicted by using the present SEM model and the experimental result cited from [28].
Table 3. Natural frequencies (Hz) of the smart beam depending on whether the PZT layer is used as an actuator or a sensor. (Note: \( n \) = total number of finite elements used in the analysis.)

<table>
<thead>
<tr>
<th>Modes</th>
<th>Actuator</th>
<th>Sensor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SEM</td>
<td>ANSYS</td>
</tr>
<tr>
<td>1</td>
<td>4.437</td>
<td>4.447</td>
</tr>
<tr>
<td>2</td>
<td>28.974</td>
<td>29.026</td>
</tr>
<tr>
<td>3</td>
<td>78.817</td>
<td>78.970</td>
</tr>
<tr>
<td>4</td>
<td>160.06</td>
<td>160.41</td>
</tr>
<tr>
<td>5</td>
<td>261.72</td>
<td>262.32</td>
</tr>
<tr>
<td>10</td>
<td>1167.7</td>
<td>1172.7</td>
</tr>
<tr>
<td>13*</td>
<td>1941.8</td>
<td>1959.3</td>
</tr>
<tr>
<td>20</td>
<td>4394.7</td>
<td>4525.1</td>
</tr>
</tbody>
</table>

* Axial modes.

Figure 7. Comparison of the flexural wave signal envelope predicted by using the present spectral element model and the experimental result from [28].

Figure 8. Comparison of experimental guided waves with those predicted by using the present spectral element model and the previous spectral element model [6].

Table 4. Natural frequencies (Hz) of the smart beam obtained by the present model, Park’s model [15] and ANSYS. (Note: \( (%) \) = % difference with respect to ANSYS.)

<table>
<thead>
<tr>
<th>Modes</th>
<th>Present model</th>
<th>Park’s model [15]</th>
<th>ANSYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.437 (0.01%)</td>
<td>4.4883 (1.14%)</td>
<td>4.4375</td>
</tr>
<tr>
<td>2</td>
<td>28.974 (0.04%)</td>
<td>31.107 (7.41%)</td>
<td>28.962</td>
</tr>
<tr>
<td>3</td>
<td>78.817 (0.06%)</td>
<td>80.848 (2.64%)</td>
<td>78.768</td>
</tr>
<tr>
<td>4</td>
<td>160.06 (0.01%)</td>
<td>168.88 (5.50%)</td>
<td>160.07</td>
</tr>
<tr>
<td>5</td>
<td>261.72 (0.05%)</td>
<td>275.51 (5.32%)</td>
<td>261.60</td>
</tr>
<tr>
<td>10</td>
<td>1167.7 (0.00%)</td>
<td>1229.1 (5.26%)</td>
<td>1167.7</td>
</tr>
<tr>
<td>13*</td>
<td>1941.8 (0.01%)</td>
<td>2086.8 (7.48%)</td>
<td>1941.6</td>
</tr>
<tr>
<td>20</td>
<td>4394.7 (0.00%)</td>
<td>4616.4 (5.04%)</td>
<td>4394.8</td>
</tr>
</tbody>
</table>

* Axial mode.

In [28], the experiment was conducted for a steel bar (dimensions: 6 mm × 6 mm × 1000 mm; \( E = 200.11 \text{ GPa}, \nu = 0.33, \rho = 7566 \text{ kg m}^{-3} \)) by applying an excitation signal of center frequency 100 kHz to the surface-bonded PZT actuator. From figure 1, the flexural wave envelope predicted by using the present SEM model is found to agree well with the experimental signal. Figure 8 represents the comparison of the guided waves predicted by using the present SEM model and the previous Bernoulli–Euler beam theory SEM model [6] with the experimentally measured signal. The previous SEM model [6] was developed by applying the Bernoulli–Euler beam theory to both the base beam and the PZT layer. An experiment has been conducted for a beam as shown in figure 5, adjusting the distance between the PZT actuator and the PZT sensor to be 600 mm and applying an excitation signal of center frequency 150 kHz. It is obvious from figure 8 that the present SEM model provides a very reliable result which is very close to the experimental result, while the previous SEM model [6] fails to provide a satisfactory result.

Finally, the natural frequencies of the example smart beam shown in figure 2 are displayed in table 4 in order to compare the present smart Timoshenko beam model with Park’s model [15]. As discussed in the last part of section 2, the present model and Park’s model have the same forms of equations of motion, but with different structural rigidities due to the application of different constitutive relations to axial and transverse bending vibrations. Table 4 shows that the results by ANSYS (2D plane stress analysis) are much closer to those by the present model, when compared with those by Park’s model [15]. This is simply due to the fact that Park’s model employs constitutive relations in an inconsistent way; that is, the plane stress-induced constitutive relations to the transverse bending vibration part and the plane stress-induced constitutive relations to the axial vibration part.

4.2. Dispersion curves

Figure 9 compares the dispersion curves for uniform metallic beams (not covered with a PZT layer) for two cases: (1) when the elementary rod theory-based beam model is used (denoted by ELT) and (2) when the Mindlin–Herrmann rod theory-based beam model is used (denoted by MHT). The ELT-based beam model and its spectral element model can be readily reduced from the MHT-based ones by removing the
terms related to the lateral contraction. In figure 9, $k_1$, $k_2$, $k_3$ and $k_4$ represent the wavenumbers for the axial (longitudinal) wave, the bending wave, the shear wave, and the lateral contraction wave, respectively. The dispersion curve for $k_4$ exists only when the MHT-based beam model is used. As the transverse bending displacement and the axial displacement are completely decoupled for the case of bare metallic beams, the dispersion curves $k_2$ and $k_3$ for both ELT-based and MHT-based beam models are found to be identical and the difference between the ELT-based and MHT-based beam models can be well observed at high frequency only from the dispersion curves $k_1$. Comparing figures 9(a) and (b) shows that the dispersion curve $k_1$ for the ELT-based beam model deviates significantly from that for the MHT-based beam model at low frequency as the base beam thickness gets increased. Figure 9 also shows that the cut-off frequencies for $k_3$ and $k_4$ tend to shift to lower frequencies as the base beam thickness is increased.

Similarly, figure 10 compares the dispersion curves for uniform smart beams fully covered with a PZT layer of thickness $h_p = 0.5$ mm when the ELT-based and MHT-based beam models are used. When compared with figure 9, one additional dispersion curve $k_5$ appears, which represents the evanescent wave in the PZT layer. As the PZT layer induces the coupling between the transverse bending displacement and the axial displacement, the differences between the dispersion curves $k_1$, $k_2$, and $k_3$ when the ELT-based beam model is used and those when the MHT-based beam model is used become larger, especially at high frequency. Comparing figures 9 and 10, it can be observed that the existence of a PZT layer in general tends to lower the cut-off frequencies of the shear wave ($k_3$) and the lateral contraction wave ($k_4$). In addition, the dispersion characteristics of the lateral contraction wave ($k_4$) and the evanescent wave in the PZT layer ($k_5$) are found to be very sensitive to the PZT layer thickness. Although the existence of a PZT layer tends to change the dispersion characteristics significantly when the PZT layer is very thick, its overall effects on the wave propagation in the metallic beam will be negligible in general provided the dimensions of the PZT layer used as an actuator or a sensor are small enough with respect to those of the base structure.

4.3. Natural frequencies and guided waves

To investigate the effects of the lateral contraction on the natural frequencies and guided wave characteristics, a cantilevered uniform metallic beam (length 600 mm, thickness 5 mm) and a uniform smart beam fully covered with a PZT layer (length 600 mm; base beam thickness 5 mm; PZT layer thickness 2 mm) are considered. The material properties used for numerical analysis are the same as those given in tables 1 and 2.

Recall that the effects of lateral contraction are taken into account in the MHT-based beam model, but not in the ELT-based beam model. Thus, in order to investigate
the effects of the lateral contraction, the natural frequencies obtained by the ELT-based and MHT-based beam models are compared in table 5. For the case of the bare metallic beam, the transverse bending displacement and the axial displacement are completely decoupled (i.e., $K_1 = K_2 = K_4 = K_6 = K_7 = m_{u\theta} = m_{\theta u} = m_{w\theta} = m_{\theta w} = 0$), while the lateral contraction is coupled to the axial displacement via the coupling rigidity $K_3$ due to the adoption of MHT. Thus, table 5 shows that the natural frequencies of the bending vibration modes obtained by the MHT-based beam model are identical to those obtained by the ELT-based beam model. However, for the case of the smart beam, coupling between the transverse bending displacement and the axial displacement will be induced due to the existence of the PZT layer. Thus, table 5 certainly shows that, for the case of the smart beam, the natural frequencies of the bending vibration modes obtained by the MHT-based beam model are slightly different from those obtained by the ELT-based beam model. As the lateral contraction is directly coupled to the axial displacement via the coupling rigidity $K_3$, table 5 also shows that the effects of lateral contraction are more significant for the axial vibration modes for both MHT-based and ELT-based beam models.

Figure 11 compares symmetric longitudinal waves predicted by using the ELT-based and MHT-based beam models with variation of the center frequency ($f_c$) of the input Morlet wavelet signal as well as the base beam thickness ($h_b$). It is obvious from figure 11 that the difference between the symmetric longitudinal waves predicted by using the
ELT-based beam model and those by the MHT-based beam model becomes more significant at higher frequency as the beam thickness gets thicker. This implies that the MHT-based beam model must be applied to thick smart beam models especially at very high frequency.

4.4. Constraint forces at the interface between the base beam and the PZT layer

Physically, the Lagrange multipliers $\lambda_1$ and $\lambda_2$ given by equation (13) represent the shear (tangential) and peeling (normal) constraint forces at the interface between the metallic base beam and the PZT layer. In order to investigate these constraint forces, a simply supported smart beam as shown in figure 12 is considered. The material properties of the smart beam are the same as those displayed in tables 1 and 2. A unit impulse of voltage is applied to the PZT layer to generate the constraint forces. For the computation of the constraint forces, hysteretic material damping is taken into account for the metallic base beam by using a complex modulus $E^* = E(1 + i\eta)$ with $\eta = 0.005$.

Time histories of the shear and peeling constraint forces at the whole interface between the metallic base beam and the PZT layer are displayed at figures 13(a) and (e), respectively. Figures 13(b)–(h) display the time histories at three specific locations: at $x = 0$ mm (middle of the PZT layer), at $x = 50$ mm, and at $x = 100$ mm (right edge of the PZT layer). It is obvious from figure 13 that both the shear and the peeling constraint forces have the largest values near to the edges of the PZT layer.
Figure 14. Wave propagation of the shear ($\lambda_1$) and peeling ($\lambda_2$) constraint forces along the interface between the metallic base beam and the PZT layer.

Figure 15. Distributions of the static shear ($\lambda_1$) and peeling ($\lambda_2$) constraint forces at the interface between the metallic base beam and the PZT layer when a static input voltage is applied to the PZT layer.

Figure 14(a) displays the wave propagation of the shear constraint force along the interface, while figure 14(b) displays the wave propagation of the peeling constraint force. As explained by equation (32), the input voltage applied at the PZT layer acts as equivalent forces and moments applied at the nodes near the left and right edges of the PZT layer. Thus, figures 14(a) and (b) clearly show that the waves of constraint forces start propagating simultaneously from the left and right edges of the PZT layer and, during their propagation, they are reflected repeatedly at the opposite edges of the PZT layer as well as at the boundaries of the smart beam.

Figure 15 displays the distributions of the static shear and peeling constraint forces generated when a static input voltage is applied to the PZT layer. As the static solutions correspond to the dynamic solution at zero frequency, for the practical computation, the dynamic constraint force at an extremely low frequency close to zero (e.g., $10^{-4}$ Hz) can be taken as the static result. It is obvious from figure 15 that peak values of the shear and peeling constraint forces (that is, stress concentrations) occur near the left and right edges of the PZT layer. The static constraint forces displayed in figure 15 are found to be very similar to those reported in [22, 23, 7].

5. Conclusions

For the successful prediction of the high frequency dynamics and guided waves generated in a metallic beam by using a PZT actuator, the spectral element model for a smart beam consisting of a metallic base beam and a surface-bonded PZT layer is developed by using the variation approach. Axial-bending-shear-lateral contraction coupled equations of motion are derived first by using Hamilton’s principle with Lagrange multipliers. The plane stress-induced constitution relations are used for both axial and transverse bending vibrations based on the Timoshenko beam theory and Mindlin–Herrmann rod theory, respectively. The spectral element model formulated from frequency-dependent dynamic shape functions derived from exact free wave solutions is verified in due course by comparing with the numerical results obtained by using the commercial FEA package ANSYS as well as with experimental results. Numerical studies conducted using the present spectral element model show the following.

(1) The natural frequencies of the smart beam have slightly larger values when the PZT layer is used as a sensor rather than an actuator.
(2) The present model provides more accurate results which are close to the results by the commercial FE analysis package ANSYS, when compared with Park’s model [15].

(3) The effect of lateral contraction is more important for axial vibration modes than for transverse bending vibration modes of the smart beam.

(4) The effect of lateral contraction on the dispersion characteristics is significant at high frequency and becomes more significant as the base beam thickness gets larger. Thus, the Mindlin–Herrmann rod theory must be adopted for high frequency axial vibration of a thick smart beam.

(5) The existence of a surface-bonded PZT layer tends to lower the cut-off frequencies of the shear and lateral contraction waves.

(6) The present spectral element model has the capability to predict the interface constraint forces by the use of Lagrange multipliers.

(7) The shear and peeling constraint forces are found to have peak values (stress concentrations) near the edges of the PZT layer.

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**Appendix A. Constitutive equations for the isotropic base beam**

The stress–strain relations for the Timoshenko beam model and the Mindlin–Herrmann rod model can be reduced from those for three-dimensional (3D) isotropic materials by using a systematic reduction approach introduced in [24].

### A.1. Timoshenko beam model: transverse bending vibration

The stress–strain relations for the transverse bending vibration of a Timoshenko beam model can be reduced from those for 3D isotropic materials by imposing the plane stress assumption $\sigma_{yy} = \tau_{xy} = \tau_{xz} = 0$ as

$$
\begin{align}
\left[ \begin{array}{c}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{array} \right] &=
\begin{bmatrix}
E & 0 & \nu E \\
0 & G & 0 \\
0 & 0 & \nu G
\end{bmatrix}
\left[ \begin{array}{c}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{array} \right]
\end{align}
$$

(A.1)

where $E$ and $G$ are the Young’s modulus and shear modulus, respectively.

### A.2. Mindlin–Herrmann rod model: axial vibration

The stress–strain relations for the axial vibration of a Mindlin–Herrmann rod model can be reduced from those for 3D isotropic materials by imposing the plane stress assumption $\sigma_{yy} = \tau_{xy} = \tau_{xz} = 0$ as

$$
\begin{align}
\left[ \begin{array}{c}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{array} \right] &=
\begin{bmatrix}
C_{11} & C_{13} & 0 \\
C_{13} & C_{13} & 0 \\
0 & 0 & C_{55}
\end{bmatrix}
\left[ \begin{array}{c}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{array} \right]
\end{align}
$$

(A.2)

where

$$
\begin{align}
C_{11} &= C_{33} = \frac{E}{1 - \nu^2} \\
C_{13} &= C_{55} = \frac{E\nu}{1 - \nu^2} \\
C_{55} &= G
\end{align}
$$

(A.3)

Here, $\nu$ is the Poisson’s ratio. The plane stress-induced constitutive relations (A.2) are used in [25, 26], while the plane strain-induced constitutive relations ($C_{11} = C_{33} = \lambda + 2\mu$, $C_{13} = \lambda$ and $C_{55} = G$, where $\lambda$ and $\mu$ are Lamé’s constraints) are used in [10, 15].

**Appendix B. Constitutive equations for the PZT layer**

The constitutive relation for the thin PZT layer subjected to a uni-axial loading can be reduced from those for 3D piezoelectric materials [20] by imposing the plane stress assumption $\sigma_{yy} = \tau_{xy} = \tau_{xz} = 0$, and $E_1 = E_2 = 0$ as

$$
\begin{align}
\left[ \begin{array}{c}
\sigma_{xx} \\
E_3
\end{array} \right] &=
\begin{bmatrix}
\bar{c}_{11}^D & -\bar{h}_{31} \\
-\bar{h}_{31} & \bar{\beta}_{33}^S
\end{bmatrix}
\left[ \begin{array}{c}
\varepsilon_{xx} \\
D_3
\end{array} \right]
\end{align}
$$

(B.1)

where

$$
\begin{align}
\bar{c}_{11}^D &= c_{11}^D - c_{12}^D c_{22}^{-1} (c_{33}^D - c_{23}^D c_{22}^{-1})^{-1} \\
\bar{h}_{31} &= h_{31} - h_{32}^D c_{22}^{-1} (h_{33} - h_{32}^D c_{22}^{-1})^{-1} \\
\bar{\beta}_{33}^S &= \beta_{33}^S - h_{32}^D c_{22}^{-1} (h_{33} - h_{32}^D c_{22}^{-1})^{-1}
\end{align}
$$

(B.2)

Here, $D_3$ is the electrical displacement and $E_3$ is the electrical field in the thickness $z$-direction, $c_{ij}^D$ ($i,j = 1–3$) are elastic stiffness coefficients, $h_{ij}$ ($i,j = 1, 2$) are piezoelectric constants, and $\beta_{33}^S$ is the dielectric constant of the piezoelectric material.

**Appendix C. Effective structural and inertia properties of the smart beam**

$$
\begin{align}
EI &= EI_b + \frac{1}{2} h_b^2 E_A p \\
E_I &= \bar{c}_{11}^D E_{Ap} + \frac{1}{2} h_b^2 E_A p \\
E_A &= C_{11} A_b + E_A p = C_{11} A_b \\
\kappa G I &= \kappa C_{55} h_b A_p = (c_{11}^D - \bar{h}_{31} \bar{\beta}_{33}^{S-1}) h_b A_p \\
E_{J} &= \frac{1}{2} h_b^2 E_p L_p \\
K_1 &= \frac{1}{2} h_b E_A p \\
K_2 &= \frac{1}{2} h_b p E_A p \\
K_3 &= C_{13} h_b p \\
K_4 &= \frac{1}{2} h_b p E_p h_b p \\
K_5 &= \frac{1}{2} h_b E_p h_b \\
K_6 &= \frac{1}{2} h_b E_p L_p \\
K_7 &= \frac{1}{2} h_b^2 E_p h_b p
\end{align}
$$

(C.1)
Appendix D. Matrix components in equation (17)

\[
\begin{align*}
X_{11} &= -k^2EA_m + m_m\omega^2 \\
X_{22} &= -k^2E_p - k^2GA_b + m_{w_p}\omega^2 \\
X_{33} &= -k^2EI - k^2GA_b + m_{\omega}\omega^2 \\
X_{44} &= -k^4E - k^2E_{GS} - EA_{\psi} + m_{w_p}\omega^2 \\
X_{12} &= -X_{21} = -ik^3K_2 \\
X_{13} &= X_{31} = k^2K_1 - m_{\omega}\omega^2 \\
X_{14} &= -X_{41} = -ik^3K_1 - ikK_3 \\
X_{23} &= -X_{32} = -ik^3K_4 + ikK_5 \\
X_{24} &= X_{42} = -k^4K_0 + m_{w_p}\omega^2 \\
X_{34} &= -X_{43} = ik^3K_7.
\end{align*}
\]

Appendix E. Matrix \(H(\omega)\) in equation (26)

\[
H(\omega) = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\rho_{w_1} & \rho_{w_2} & \rho_{w_3} & \cdots & \rho_{w_{10}} \\
\rho_{\psi_1} & \rho_{\psi_2} & \rho_{\psi_3} & \cdots & \rho_{\psi_{10}} \\
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \cdots & \varepsilon_{10} \\
\rho_{w_1}\varepsilon_1 & \rho_{w_2}\varepsilon_2 & \rho_{w_3}\varepsilon_3 & \cdots & \rho_{w_{10}}\varepsilon_{10} \\
\rho_{\psi_1}\varepsilon_1 & \rho_{\psi_2}\varepsilon_2 & \rho_{\psi_3}\varepsilon_3 & \cdots & \rho_{\psi_{10}}\varepsilon_{10} \\
\rho_{p_1}\varepsilon_1 & \rho_{p_2}\varepsilon_2 & \rho_{p_3}\varepsilon_3 & \cdots & \rho_{p_{10}}\varepsilon_{10}
\end{bmatrix}
\]

where

\[
r_{pj} = -i\left(r_{w_j} + \frac{1}{2}h_b r_{\psi j}\right)k_j \\
e_j = e^{-ik_jl} \quad (j = 1, 2, 3, \ldots, 10).
\]

Appendix F. Matrix \(D(\omega)\) in equation (34)

\[
D(\omega) = -EA_{\psi}KEK - EJ_{\psi}KEK_{\psi} + EA_{\psi}R_{\psi}ER_{\psi} \\
+ E\rho_{w_1}K^2EK^2R_{w} + k^2GA_b(R_{w}ER_{\psi}) \\
- R_{\psi}KEK_{\psi} - k^2GIR_{\psi}KEK_{\psi} \\
+ E\rho_{\psi}K^2E^2R_{\psi} + K_b(R_{\psi}K^2EK^2R_{w} \\
+ R_{\psi}K^2EK^2R_{\psi} + iK_3(R_{\psi}K^2EK_{\psi}) \\
+ R_{\psi}KEK^2R_{\psi} + iK_4(R_{w}K^2EK_{\psi} + R_{\psi}KEK_{\psi} \\
- K_3(R_{w}KEK_{\psi} + R_{\psi}KEK_{\psi}) - iK_5(K_3(KEK_{\psi} \\
+ R_{\psi}KE) + iK_4(R_{w}K^2EK_{\psi} + R_{\psi}KEK_{\psi} \\
- KEK^2R_{\psi} - R_{\psi}K^2EK) + iK_3(R_{w}KEK_{\psi} \\
+ R_{\psi}KEK_{\psi} + \frac{i}{2}(m_{\omega}\rho ER_{\psi} + m_{\omega}{R_{w}ER_{\psi}} \\
+ m_{w_p}R_{\psi}ER_{\psi} - m_{w_p}R_{w}ER_{\psi} \\
- m_{w_p}R_{\psi}ER_{\psi}) \\
- (m_{\omega}\rho ER_{\psi} + m_{w_p}R_{\psi}ER_{\psi}]
\]

where

\[
K = \text{diag}[k_j] \\
k = \text{diag}[k_j] \\
E(\omega) = [E_{mn}(\omega)] \\
(m, n = 1, 2, 3, \ldots, 10)
\]

with

\[
E_{mn}(\omega) = \begin{cases}
\frac{i}{k_m + k_n}[e^{-i(k_m+k_n)l} - 1] & \text{if } k_m + k_n \neq 0 \\
n & \text{if } k_m + k_n = 0.
\end{cases}
\]

Appendix G. Matrix \(\hat{S}\) in equation (38)

\[
\hat{S} = \frac{bh_p^2p_{33}^2}{l} \begin{bmatrix}
\Omega & -\Omega \\
-\Omega & \Omega
\end{bmatrix}
\]

where

\[
\Omega = \begin{bmatrix}
0 & -\frac{1}{2}h_b & 0 & -\frac{1}{2}h_p \\
0 & 0 & 0 & 0 \\
-\frac{1}{2}h_b & 0 & \frac{1}{2}h_p & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

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[28] Rucka M 2010 Experimental and numerical studies of guided wave damage detection in bars with structural discontinuities Arch. Appl. Mech. 80 1371–90 (Fig. 6a, p. 1380)