Nonclassical measure of nonclassical properties

Kisik Kim
Department of Physics, Inha University, Inchon, Korea 402-751
and Department of Physics, University of Oregon, Eugene, Oregon 97403-1274
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The nonclassical measures of nonclassical properties of radiation fields are proposed in parallel with those of the nonclassical states. It is shown that the nonclassical measures of sub-Poissonian photon statistics and squeezing are bounded below 1/2 and their physical implications are discussed in comparison with the nonclassical measures of states. [S1050-2947(98)00611-8]

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I. INTRODUCTION

Since Wigner introduced a distribution function $W$, now known as the Wigner function, to characterize the state of a quantum system in phase space [1], other phase-space densities have been proposed and their connections have been investigated [2,3]. Using these phase-space densities, one can calculate quantum-mechanical expectation values of operators in a way completely analogous to the way an average is calculated in classical probability theory. The phase-space densities, however, do not meet all the properties that a classical probability density satisfies. They sometimes become negative or more singular than a $\delta$ function and then the corresponding states are purely quantum-mechanical states without classical analogs. In this sense, phase-space densities are called quasiprobability densities.

In quantum optics, operators are often expressed in terms of the photon creation and annihilation operators and the different phase space densities originate from different orderings for the creation and annihilation operators of the operator whose expectation value is to be evaluated. Among the various orderings, a particular set of orderings, called $s$ orderings, can be parametrized by a real parameter $s$ ($-1 \leq s \leq 1$) [3]. The widely used phase-space densities such as the Glauber-Sudarshan $P$ function ($s=1$) [4,5], the Wigner $W$ function ($s=0$) [1], and the Husimi $Q$ function ($s=-1$) belong to this category. The phase-space densities corresponding to $s$ orderings can be obtained as convolution transforms of the $P$ function with appropriate Gaussian functions and hence any $s$-ordered phase-space density is smoother than the $P$ function itself. Recently, the measure of nonclassicality of a state has been introduced through the regularization process of the $P$ function [6].

In this paper we consider the measure of nonclassicality of a nonclassical property. The nonclassical states may show various nonclassical properties; however, these nonclassical properties are sometimes considered incompatible. For instance, squeezing and sub-Poissonian photon statistics depend on two different marginal phase-space densities [7]. It may be possible to have a particular nonclassical property washed out before the complete regularization process of the $P$ function is made. This indeed happens and we consider two typical nonclassical properties: squeezing and sub-Poissonian photon statistics.

II. PHASE-SPACE DENSITIES AND SUPERPOSITION

The phase-space density in the $s$ ordering, denoted by $\phi_s$ ($-1 \leq s \leq 1$), is defined by

$$\phi_s(v) = \frac{2}{\pi(1-s)} \int e^{-2|v-v'|^2/(1-s)} \phi_1(v')d^2v'.$$

(2.1)

$\phi_1$ is called the $P$ function and plays a special role in connection with the criterion of nonclassical states since it is possibly most singular among all the $s$-ordered phase-space densities.

Now let us consider the superposition of two independent radiation fields through a beam splitter (Fig. 1). The beam splitter has two input ports, denoted by the associated modes $\hat{a}$ and $\hat{b}$, and two output ports $\hat{c}$ and $\hat{d}$. Two independent beams are sent to the two input ports and one of the beams emerging from the beam splitter, say $\hat{c}$, is examined. Suppose we obtain the density operator $\hat{\rho}_1$ with the input port $\hat{b}$ blocked and $\hat{\rho}_2$ with the input port $\hat{a}$ blocked, respectively.

The density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ may be written as

$$\hat{\rho}_1 = \int P_1(v)|v\rangle\langle v|d^2v,$$

(2.2)

$$\hat{\rho}_2 = \int P_2(v)|v\rangle\langle v|d^2v.$$

(2.3)

FIG. 1. Illustration of the notations for the beam splitter.
As we show in the Appendix, when we open two ports together, we obtain the superposition field, the $P$ function of which is given by the convolution of the two individual $P$ functions

$$P_{12}(v) = \int P_2(v-v')P_1(v')d^2v',$$  \hspace{1cm} (2.4)

where the integration is over the entire complex $v'$ plane and

$$d^2v' = d(\text{Re} v')d(\text{Im} v').$$ \hspace{1cm} (2.5)

Note that $\hat{\rho}_1$ and $\hat{\rho}_2$ are not the density operators of the incident beams, but those of transmitted and reflected beams via the beam splitter.

Comparing Eqs. (2.1) and (2.4), we may regard $\phi_1$ as the $P$ function of the superposition field of the given radiation field with the radiation field whose $P$ function is given by

$$P_2(v) = \frac{2}{\pi(1-s)}e^{-2|v|^2/(1-s)}.$$ \hspace{1cm} (2.6)

It turns out that this $P$ function corresponds to the chaotic light of the average photon number $(1-s)/2$. We introduce a new parameter $\tau$ as

$$\tau = \frac{1-s}{2}. \hspace{1cm} (2.7)

Then $\tau$ is the average photon number of the chaotic light and it takes on a real value between 0 and 1.

The measure of nonclassicality of a state is defined by the minimum value of $\tau$, denoted by $\tau_m$, above which $\phi_1$ becomes an admissible function in the classical probability theory. For a coherent state $\tau_m = 0$, for a Fock state $\tau_m = 1$, and for a squeezed vacuum state $0 < \tau_m < 1/2$ depending on the degree of squeezing. In view of the interpretation in terms of superposition, we may say that the measure of nonclassicality is the minimum average photon number of the chaotic light to destroy all the nonclassical properties of the given radiation field. We emphasize that the above statement is for the entire destruction of the nonclassical properties. If we consider only one particular nonclassical property, we may not need all the $\tau_m$ photons to destroy it.

III. SUPERPOSITION AND OPERATOR ORDERING

Using the phase-space densities, we can calculate the quantum-mechanical expectation value of the operator $\hat{O}(\hat{a},\hat{a}^\dagger)$ for the given density operator $\hat{\rho}(\hat{a},\hat{a}^\dagger)$ as

$$\langle \hat{O} \rangle = \text{Tr}(\hat{\rho}\hat{O}) = \int \phi_1(v)\hat{O}_1(v,v^*)d^2v,$$ \hspace{1cm} (3.1)

where $\hat{O}_1(v,v^*)$ is a $c$-number function obtained by replacing the operators $\hat{a}$ and $\hat{a}^\dagger$ by $v$ and $v^*$, respectively, in the $s$-ordered expression $\hat{O}_1(\hat{a},\hat{a}^\dagger)$ of $\hat{O}(\hat{a},\hat{a}^\dagger)$. The normal ordering corresponds to $s = 1$, which the $P$ function is associated with.

Since we may regard $\phi_1$ as the $P$ function of the superposition field with the chaotic light of the average photon number $\tau = (1-s)/2$, we can consider Eq. (3.1) as the quantum-mechanical expectation value of the operator $\hat{O}$, whose normally ordered expression is given by $\hat{O}_1$ for the superposition field [8]. When we denote the density operator of the superposition field by $\hat{\rho}'$, we then have the series of equalities

$$\langle \hat{O}_1 \rangle_{\rho} = \int \phi_1(v)\hat{O}_1(v,v^*)d^2v = \int \phi_1(v)\hat{O}_1(v,v^*)d^2v = \langle \hat{O}_1 \rangle_{\rho'}.$$ \hspace{1cm} (3.2)

It is the main task of the present paper to find the operator $\hat{O}_1$. Fortunately, all the necessary formulas can be found in the existing literatures and hence we summarize them briefly [3].

When we define the $s$-ordered displacement operator $D(s,a)$ as

$$D(s,a) = D(s)e^{s|a|^2/2},$$ \hspace{1cm} (3.3)

we then obtain the $s$-ordered product $\{(\hat{a}^\dagger)^s\hat{a}^m\}_s$ as

$$\{(\hat{a}^\dagger)^s\hat{a}^m\}_s = \frac{\partial^{n+m}D(s,a)}{\partial \alpha^n\partial (-\alpha^*)^m}_{_{a=0}}.$$ \hspace{1cm} (3.5)

The connection formula between two $s$-ordered expressions is given by

$$\{(\hat{a}^\dagger)^s\hat{a}^m\}_s = \sum_{k=0}^{(n,m)} k! \left( \begin{array}{c} n \cr k \end{array} \right) \left( \begin{array}{c} m \cr k \end{array} \right) \{(\hat{a}^\dagger)^{s-k}\hat{a}^{m-k}\}_s.$$ \hspace{1cm} (3.6)

where $(n,m)$ denotes the smaller of the integers $n$ and $m$.

If we have the $s$-ordered expression $\hat{O}_1$ of the operator $\hat{O}$ in a form of a power series

$$\hat{O} = \hat{O}_1 = \sum_{n,m} \hat{O}_{nm}\{(\hat{a}^\dagger)^n\hat{a}^m\}_s,$$ \hspace{1cm} (3.7)

$\hat{O}'$ can be obtained by replacing $\{(\hat{a}^\dagger)^n\hat{a}^m\}_s$ in Eq. (3.7) by $\{(\hat{a}^\dagger)^n\hat{a}^m\}_1$.

As a simple example, let us consider the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. Applying Eq. (3.5) or (3.6), we have

$$\hat{n} = \{\hat{n}\}_s = \frac{1-s}{2}. \hspace{1cm} (3.8)

Equation (3.2) now reads

$$\text{Tr}(\hat{\rho}\hat{n}) = \text{Tr}(\hat{\rho}'\hat{n}')$$ \hspace{1cm} (3.9)

and $\hat{n}'$ is given by

$$\hat{n}' = \hat{n} - \frac{1-s}{2}. \hspace{1cm} (3.10)$$
From Eq. (3.8) we have
\[
\langle \hat{n} \rangle_\rho = \langle \hat{n} \rangle_\rho^' = \langle \hat{n} \rangle_\rho - \frac{1 - s}{2}.
\] (3.11)

Consequently,
\[
\langle \hat{n} \rangle_\rho^' = \langle \hat{n} \rangle_\rho + \tau.
\] (3.12)

Equation (3.12) implies that the average photon number in the superposition field increases by \( \tau \) and this result is expected.

**IV. NONCLASSICAL MEASURES OF SQUEEZING AND SUB-POISSONIAN PHOTON STATISTICS**

Now let us consider two typical nonclassical properties of light: sub-Poissonian photon statistics and squeezing. Using Eq. (3.5), we obtain the \( s \)-ordered expression of \( \hat{n}^2 \) as
\[
\hat{n}^2 = \{\hat{n}^2\}_s + (2s - 1)\{\hat{n}\}_1 + \frac{s(s - 1)}{2}
\] (4.1)

and hence
\[
\langle \hat{n}^2 \rangle_\rho^' = \{\hat{n}^2\}_s + (2s - 1)\{\hat{n}\}_1 + \frac{s(s - 1)}{2}.
\] (4.2)

Using the fact
\[
\hat{n}^2 = \{\hat{n}^2\}_1 + \{\hat{n}\}_1,
\] (4.3)
\[
\hat{n} = \{\hat{n}\}_1,
\] (4.4)

Eq. (4.2) can be rewritten as
\[
\langle \hat{n}^2 \rangle_\rho^' = \langle \hat{n}^2 \rangle_\rho + 2(s - 1)\langle \hat{n} \rangle_\rho + \frac{s(s - 1)}{2}.
\] (4.5)

On substituting Eq. (4.5) into Eq. (3.2), we have
\[
\langle \hat{n}^2 \rangle_\rho = \langle \hat{n}^2 \rangle_\rho^' + 2(s - 1)\langle \hat{n} \rangle_\rho + \frac{s(s - 1)}{2}.
\] (4.6)

and, utilizing Eq. (3.12), we finally have
\[
\langle \hat{n}^2 \rangle_\rho^' = \langle \hat{n}^2 \rangle_\rho - 2(s - 1)\langle \hat{n} \rangle_\rho + \frac{(s - 1)(s - 2)}{2}.
\] (4.7)

Equations (4.6) and (4.7) allow us to calculate
\[
\langle (\Delta \hat{n})^2 \rangle_\rho = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2.
\] (4.8)

After a straightforward calculation, we have
\[
\langle (\Delta \hat{n})^2 \rangle_\rho^' = \langle (\Delta \hat{n})^2 \rangle_\rho - (s - 1)\langle \hat{n} \rangle_\rho + \frac{(1 - s)(3 - s)}{4}.
\] (4.9)

The sub-Poissonian property is often characterized by the number \( Q \) defined by [9]

\[
Q = \frac{\langle (\Delta \hat{n})^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle}.
\] (4.10)

For our present purpose, we are interested in the possible crossover from sub-Poissonian to super-Poissonian and hence interested in the sign of \( Q \). To see this, we look at the expression
\[
\langle (\Delta \hat{n})^2 \rangle_\rho^' - \langle \hat{n} \rangle_\rho^' = [\langle (\Delta \hat{n})^2 \rangle_\rho - \langle \hat{n} \rangle_\rho] + 2\tau\langle \hat{n} \rangle_\rho + \tau^2.
\] (4.11)

Suppose the initial field shows a sub-Poissonian photon statistics and the associated \( Q \) value is \( Q_0 \). Then the minimum average photon number in the chaotic light can be calculated, in particular, when we consider the case in which \( \langle \hat{n} \rangle \gg 1 \). We then have
\[
\tau_m = -\frac{Q_0}{2}.
\] (4.12)

One can easily find
\[
(\text{Re} \, \hat{a})^2 = (\text{Re} \, \hat{a})^2
\] (4.13)

and, on using Eq. (3.8),
\[
[\langle (\text{Re} \, \hat{a})^2 \rangle]^' = (\text{Re} \, \hat{a})^2 + \frac{s - 1}{4}.
\] (4.14)

Following the series of equalities of Eq. (3.2), we have
\[
\langle (\Delta \text{ Re} \, \hat{a})^2 \rangle_\rho^' = \langle (\Delta \text{ Re} \, \hat{a})^2 \rangle_\rho + \frac{1 - s}{4}.
\] (4.15)

The squeezing occurs when
\[
\langle (\Delta \text{ Re} \, \hat{a})^2 \rangle < \frac{1}{4}.
\] (4.16)

The degree of squeezing is defined, in parallel with that of sub-Poissonian photon statistics, as the amount of deviation from the fluctuation of the coherent state, i.e.,
\[
\beta = 4\langle (\Delta \text{ Re} \, \hat{a})^2 \rangle - 1.
\] (4.17)

Then the negative value of \( \beta \) implies squeezing. Now, if the original field shows squeezing and the associated \( \beta \) value is \( \beta_0 \), then the minimum average photon number in the chaotic light is given by
\[
\tau_m = -\frac{\beta_0}{2},
\] (4.18)

to eliminate the squeezing property upon superposition.
V. DISCUSSION AND CONCLUSION

By definitions [Eqs. (4.10) and (4.18)], the maximum sub-Poissonian photon statistics and the maximum squeezing occur when $Q = -1$ and $\beta = -1$, respectively, and hence the nonclassical measures for both of these cases cannot exceed the value 1/2. This result is somewhat interesting in comparison with the measure of nonclassicality of the nonclassical state.

The typical nonclassical state showing sub-Poissonian photon statistics is a Fock state and, for any Fock state, $Q = -1$. While the measure of nonclassicality of any Fock state is unity, the measure of nonclassicality of the sub-Poissonian property is just 1/2. In other words, one photon is necessary to destroy all the nonclassical properties of a Fock state, while only 1/2 photon is needed to eliminate the sub-Poissonian character. The sub-Poissonian character is only a partial indication for a Fock state to be nonclassical.

On the other hand, for the squeezed vacuum state, which is generated from the vacuum state by the squeezing operator [10]

$$\hat{S}(\eta) = \exp \left( \frac{1}{2} \eta (\hat{a}^\dagger)^2 - \frac{1}{2} \eta^* \hat{a}^2 \right),$$

(5.1)

where $\eta = re^{i\theta}$, the degree of squeezing is given by

$$\beta = -(1 - e^{-2r}).$$

(5.2)

From Eqs. (4.16) and (5.2) we have

$$\tau_m = \frac{1}{2} (1 - e^{-2r}).$$

(5.3)

This $\tau_m$ is equal to the measure of nonclassicality of the squeezed vacuum state itself [11]. This implies, for the squeezed vacuum state, one need not consider any other nonclassical character than the squeezing itself to test its nonclassicality.

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APPENDIX: SUPERPOSITION THROUGH A BEAM SPLITTER

We consider the beam splitter illustrated in Fig. 1. Let us assume that the density operator at the input of the beam splitter has been given a diagonal coherent-state representation in the two input mode amplitudes $v_1$ and $v_2$, in the usual form

$$\hat{\rho}_m = \int \phi_{in}(v_1,v_2) |v_1,v_2\rangle \langle v_1,v_2| d^2v_1 d^2v_2. \quad (A1)$$

Similarly, the density operator at the output can be written in the diagonal form in terms of the two complex output mode amplitudes $w_1$ and $w_2$ as

$$\hat{\rho}_o = \int \phi_{out}(w_1,w_2) |w_1,w_2\rangle \langle w_1,w_2| d^2w_1 d^2w_2. \quad (A2)$$

For the symmetric beam splitter, the input mode and the output mode operators are related as

$$\hat{c} = \hat{a} + r \hat{b}, \quad (A3)$$

$$\hat{d} = r \hat{a} + \hat{b}, \quad (A4)$$

where $t$ and $r$ are transmittivity and reflectivity of the beam splitter, and they satisfy the reciprocity relations [12]

$$|t|^2 + |r|^2 = 1, \quad (A5)$$

$$tr^* + tr r = 0. \quad (A6)$$

The output phase-space density can be written, in terms of the input phase-space density, as [13]

$$\phi_{out}(w_1,w_2) = \phi_{in} \left( \frac{-tw_1 + rw_2}{r^2 - t^2}, \frac{rw_1 - tw_2}{r^2 - t^2} \right). \quad (A7)$$

Now we assume that the two input fields are independent. Then the input density operator is given by the tensor product of two individual density operators as

$$\hat{\rho}_m = \hat{\rho}_a \otimes \hat{\rho}_b \quad (A8)$$

and the corresponding phase-space density can be expressed as

$$\phi_{in}(v_1,v_2) = \phi_a(v_1) \phi_b(v_2). \quad (A9)$$

The two output fields, however, are not independent and a certain amount of correlation appears. Nevertheless, we may have the two reduced density operators and the corresponding reduced phase-space densities. The reduced density operator associated with the mode $\hat{c}$ can be obtained by taking the partial trace on $\hat{\rho}_o$ over the mode $\hat{d}$ and vice versa:

$$\hat{\rho}_c = Tr_d \hat{\rho}_o, \quad (A10)$$

$$\hat{\rho}_d = Tr_c \hat{\rho}_o. \quad (A11)$$

When we apply Eqs. (A10) and (A11) to Eq. (A2) we have

$$\phi_c(w_1) = \int \phi_a \left( \frac{-tw_1 + rw_2}{r^2 - t^2} \right) \phi_b \left( \frac{rw_1 - tw_2}{r^2 - t^2} \right) d^2w_2, \quad (A12)$$

$$\phi_d(w_2) = \int \phi_a \left( \frac{-tw_1 + rw_2}{r^2 - t^2} \right) \phi_b \left( \frac{rw_1 - tw_2}{r^2 - t^2} \right) d^2w_1. \quad (A13)$$

Since the two reduced density operators are of the same form, from now on we consider $\hat{\rho}_c$ only, which is given by

$$\hat{\rho}_c = \int \phi_a \left( \frac{-tw_1 + rw_2}{r^2 - t^2} \right) \phi_b \left( \frac{rw_1 - tw_2}{r^2 - t^2} \right) |w_1\rangle \langle w_1| d^2w_1 d^2w_2. \quad (A14)$$
On using the change of variables
\[\alpha = \frac{-t w_1 + r w_2}{r^2 - r^2},\]  
\[\beta = \frac{r w_1 - t w_2}{r^2 - r^2},\]  
Eq. (A14) reduces to
\[\hat{\rho}_c = \int \phi_a(\alpha) \phi_b(\beta) |t \alpha + r \beta \rangle \langle t \alpha + r \beta| d^2 \alpha d^2 \beta.\]  
If we make a further trivial change of variables
\[\alpha' = t \alpha,\]  
\[\beta' = r \beta,\]
we then have an alternative expression for \(\hat{\rho}_c:\)
\[\hat{\rho}_c = \frac{1}{|t|^2 |r|^2} \int \phi_a(\alpha') |r \beta' \rangle \langle \beta'| d^2 \alpha' d^2 \beta'.\]  
Defining the new quasi-probability densities
\[\phi_a(\alpha') = \frac{1}{|t|^2} \phi_a(\alpha/t),\]  
\[\phi_b(\beta') = \frac{1}{|r|^2} \phi_b(\beta/r),\]
we have
\[\hat{\rho}_c = \int \phi_a(\alpha') \phi_b(\beta') |\alpha' + \beta' \rangle \langle \alpha' + \beta'| d^2 \alpha' d^2 \beta'.\]  
From Eq. (A23) we find the quasi-probability density of \(\hat{\rho}_c:\)
\[\phi_c(v) = \int \phi_a(v - v') \phi_b(v') d^2 v'.\]  
Equation (A24) has exactly the same form as the superposition formula in Eq. (2.4) with \(\phi_c, \phi'_a,\) and \(\phi'_b\) replaced by \(P_{12}, P_{11},\) and \(P_2.\) Notice, however, that the participating quasi-probability densities are not those of the incoming beams; instead, they are given by Eqs. (A21) and (A22).

Equation (A24) may not be applied to an arbitrary pair of individual \(P\) functions and there are some restrictions for the resultant \(P\) function to represent a physical state [14]. In this superposition through the beam splitter, one of the reciprocity relations (A5) guarantees the physical validity of Eq. (A24).

The coherent state and the thermal state have a special meaning in connection with Eq. (A21) or (A22). Since \(P\) functions of the coherent state and the thermal state are given by
\[\phi_{\text{coherent}}(\alpha) = \delta(\alpha - \alpha_0),\]  
\[\phi_{\text{thermal}}(\alpha) = \frac{1}{\pi N} e^{-|\alpha|^2/N},\]
where \(|\alpha_0|^2\) and \(N\) designate the average photon numbers, applying Eq. (A21) to \(\phi_{\text{coherent}}\) and to \(\phi_{\text{thermal}},\) we have
\[\phi'_{\text{coherent}}(\alpha') = \delta(\alpha' - \tau \alpha_0),\]  
\[\phi'_{\text{thermal}}(\alpha') = \frac{1}{\pi |t|^2 |r|^2} e^{-|\alpha'|^2/2N}.\]  
From Eqs. (A27) and (A28) we recognize that, for these two special cases, the transmitted and the reflected beams possess the same statistical properties as the incident beam except the photon number reduction.

The situation to which the superposition formulas (2.4) and (2.6) apply can be obtained by letting the test beam in the input port \(\alpha\) and the thermal beam in the input port \(\beta.\) What we observe in this scheme is, in general, not the properties of the test beam itself but those of the transmitted part of the test beam through the beam splitter. The properties of test beam itself can, however, be obtained in the limit where \(|r|^2 \rightarrow 0, N \rightarrow \infty,\) and \(|r|^2 N \rightarrow (1 - s) 2 = \tau.\)