理学博士学位請求論文

(0,1)-行列の永久に 関する 確率論的 極限
Probabilistic Limit about Permanent of (0,1)-matrices

1985年 2月

仁荷大 学校 大学院

数学科 (應用数学専攻)

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Probabilistic Limit about Permanent of (0,1)-matrices

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謝辞

이 논문이 이루어지기까지, 지도해 주신 냉 것님께 깊은 감사를 드리며, 검토와 더불어 본 논문의 작성에 기여해 주신 현 김대 오 시장님께 특별히 감사 드립니다.

그리고 검토를 받아주신 김예 선, 김학만 선, 김봉선 선생님들께도 심심한 감사를 드리며, 본 논문의 수용을 받아주신 교육과학상 신 중앙에 깊은 감사의 표시를 드립니다.

 언제나 기도와 기쁨적인 보상으로서, 포기와 세 자녀들에 감사 하며, 날 기도로서 지리적 염려해 주신 어머님께 이 봉양을 바칩니다.
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(0, 1)-행렬의 퍼머넷에 관한 확률론의極限

安位鍾

N개의 1 을 갖는 모든 \( n \times m (0,1) \)-행렬의集合을 \( M(n,m,N) \)이라 한다. 그리고, \( P \) 를 \( M(n,m,N) \) 上의 一様確率이라 한다.

즉, \( \binom{n^m}{N} \) 個의 各元素는 같은 確率 \( \binom{n^m}{N}^{-1} \) 로 선택되어질 수 있다고 하자.

그러면 행렬 \( \omega \subseteq M(n,m,N) \)의 퍼머넷을 \( M(n,m,N) \) 上에 定義된 確率變數로 생각할 수 있다. 이 때, 이 행렬의 퍼머넷을 Per(\( \omega \))로 나타낸다.

본 論文에서는 퍼머넷 Per(\( \omega \))의第一次積率과 二次積率을 유도하고,

\[
m = \beta n, \quad \beta \geq 1, \quad \frac{N(n)}{n^{3/2}} \to \infty
\]

일 때, 이를 퍼머넷의 一次, 二次積率의 近似的 推定値로 matching problem의極限分布를 이용하여 구하였다.

그래서 一次積率의 值이 二次積率의 近似的으로 같다는 것을 알게 되었고, 이로서 \( M(n,m,N) \)에 있어서

\[
N \to \infty, \quad n \to \infty, \quad \frac{N}{n^{3/2}} \to 0
\]

일 때,

\[
\frac{\text{Per}(\omega)}{E(\text{Per}(\omega))} \to 1 \text{ in probability}
\]

이러한 結論을 얻게 되었다. 이 정리는 거의 모든 行列은 위의 條件下에 近似的으로 같은 퍼머넷을 갖는다는 것을 意味한다.

위 결 또는 P. E. O’neil에 依하여 \( n = m \) 일 경우에 證明되었으며, 따라서 이는 \( n \leq m \)인 경우까지 확장된 定理이다.
다음으로, \( N = \frac{1}{2} - n \) 일 때, 피머넷트가 0인 행렬의 집합의 확률은 적어도 하나의

\[
\text{제0행을 가지고거나 적어도 } m - n + 1 \text{개의 0행을 가진 행렬의 집합의 확률과}\n\]

유사한 것으로 간주하는 것을 증명하였다.

이 정리는 피머넷트가 0인 거의 모든 행렬은 적어도 하나의 0행을 가지고거나 적어도

\( m - n + 1 \)개의 0행을 가진다는 것을 의미한다.
PROBABILISTIC LIMIT ABOUT PERMANENT OF 
(0, 1) - MATRICES

Wi Chong Ahn

Abstract

Let $M(n, m, N)$ be the class of all $n \times m (0, 1)$ - matrices with a given number $N$ of ones. Let $P$ be a uniform distribution on $M(n, m, N)$, i.e. so that each of the \( \binom{nm}{N} \) elements of $M(n, m, N)$ has the same probability \( \binom{nm}{N}^{-1} \) to be chosen. Then permanent of matrix $\omega \in M(n, m, N)$, which is denoted by $\text{Per}(\omega)$, may be considered as a random variable on $M(n, m, N)$.

In this dissertation, the first and the second moment of $\text{Per}(\omega)$ is derived. When $m = \beta n$, $\beta \geq 1$ and $N(n) / (n^{3/2}) \to \infty$, asymptotic estimates of these quantities are found by the limiting distribution of generalized matching problem.

It turns out that the square of the first moment is asymptotic to the second moment, so we conclude that almost all matrices have asymptotically the same permanent which is the average of $\text{Per}(\omega)$.

The above theorem was proved by P. E. O'Neil in the case of $n = m$.

Next we show that, when $N = \frac{1}{2}nm$, the probability of the set of matrices whose permanent is equal to 0 is asymptotic to the probability of the set of matrices which have at least one zero row or at least $m-n+1$ zero columns.

The above statement means that almost all matrices whose permanent is equal to 0 have at least one zero row or at least $m-n+1$ zero columns.
CHAPTER 1. INTRODUCTION

The permanent of $(0, 1)$-matrix plays an important part in combinatorics, graph theory and probability theory. Some probabilistic results concerning $(0, 1)$-matrix have been investigated by P. Erdős and A. Renyi [5], P. O'neil [8, 9].

In this dissertation the first and second moments of permanent of $n \times m$ $(0, 1)$-matrices are derived and asymptotic estimates of these quantities are found. It turns out that the square of the first moment is asymptotic to the second moment, so we conclude that almost all matrices have asymptotically the same permanent. This type of results is proved by P. O'neil [9] when $n = m$.

Next we show that the probability of the set of matrices whose permanent are equal to 0 is asymptotic to the probability of the set of matrices which has at least one zero row or at least $m - n + 1$ zero columns. The above statement means that almost all matrices whose permanent are equal to 0 have at least one zero row or at least $m - n + 1$ zero columns.

In chapter 2, we give the definitions of basic notion and its related results.

In chapter 3, we will prove our results.
CHAPTER 2. PRELIMINARIES

2.1. Permanent

The permanent appears repeatedly in the literature of combinatorics in connection with certain enumeration and extremal problem.

Let $A = (a_{ij})$ be an $n \times m$ matrix ($n \leq m$). The permanent of $A$, written $\text{Per}(A)$, is defined by

$$\text{Per}(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where the summation extends over all one-one function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

Thus $\text{Per}(A)$ consists of $m^n$ summands.

For example, if

$$A = \begin{bmatrix} 3 & 2 & 4 \\ \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ -1 & 2 & -2 \\ \end{bmatrix}$$

then $\text{Per}(A) = 9$, $\text{Per}(B) = 44$ and $\text{Per}(C) = 18$.

If $n = m$, then the terms in $\text{Per}(A)$ are, apart from the sign, just the terms in the expansion of $\det(A)$. 

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Let us have some examples related to the permanent.

If \( D \) is the \( n \)-square matrix with 0's on the main diagonal, 1's elsewhere, the \( \text{Per}(D) \) is a count of the total number of permutations with no fixed points of \( 1, 2, \ldots, n \).

This number is known as

\[
n! \left( \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \right)
\]

For systemic exposition see Minc's book "Permanent"[7].

\[2.2 \ (0, 1)-\text{matrices}\]

Matrices all of whose entries are either 0 or 1—that is, \((0,1)\)-matrices—play an important part in linear algebra, combinatorics and graph theory.

Let \( x_1, x_2, \ldots, x_n \) be subsets, not necessarily distinct, of an \( m \)-set \( S = \{ x_1, x_2, \ldots, x_m \} \).

Let \( A = (a_{ij}) \) be the \((0,1)\)-matrices whose entries are defined as follows:

\[
a_{ij} = \begin{cases} 
1 & \text{if } x_j \subseteq x_i \\
0 & \text{if } x_j \not\subseteq x_i 
\end{cases}
\]

The matrix \( A \) is called the incidence matrix for the subsets \( x_1, x_2, \ldots, x_n \) of the \( m \)-set \( S \).

Let \( x_1, x_2, \ldots, x_n \) be the subsets of an \( m \)-set \( S \).
The family \( x_1, x_2, \ldots, x_n \) is said to have a transversal if there exist \( n \) distinct elements \( x_{k_1}, \ldots, x_{k_n} \) such that

\[
x_{k_i} \subseteq x_i, \ i = 1, 2, \ldots, n
\]
An interpretation of transversality is as follows.

Think of elements \( x_i \in S \) as workers. Suppose that there are \( n \) types of jobs \( J_1, J_2, \ldots, J_n \). Introduce \( x_i \) as the set of workers who can perform the job \( J_i \). Then \( x_1, x_2, \ldots, x_n \) has a transversal if and only if there exists an assignment of worker such a way that all jobs may be done.

It is known that the family \( x_1, x_2, \ldots, x_n \) has a transversal if and only if \( \text{Per}(A) > 0 \), where \( A \) is the incidence matrix of subsets \( x_1, x_2, \ldots, x_n \) of an \( m \)-set \( S \).

The most fundamental result in the combinatorial matrix theory and the theory of the permanent of non-negative matrices is the so-called Frobenius-König Theorem. We will quote this theorem several times for our main results.

**THEOREM** The subsets \( x_1, x_2, \ldots, x_n \) do not have a transversal if and only if there is a \( r \times s \) zero submatrix of \( A \) such that \( r+s \geq m+1 \), where \( A \) is the incidence matrix of subsets \( x_1, x_2, \ldots, x_n \) of \( m \)-set \( S \).

We can restate Frobenius-König Theorem as follows:

Let \( A \) be any \( n \times m \) \((0, 1)\)-matrix. \( \text{Per}(A) = 0 \) if and only if there is a \( r \times s \) zero submatrix of \( A \) such that \( r+s \geq m+1 \).

Let \( M(n, m, N) \) denote the set of all \( n \times m \) matrices among the elements of which there are exactly \( N \) elements \((N \leq nm)\) equal to 1, all the other elements are equal to 0. The set \( M(n, m, N) \) contains clearly \( \binom{nm}{N} \) such matrices. We may regard an element \( A \) of \( M(n, m, N) \) as a graph containing \( n \) red and \( m \) blue vertices and \( N \) edges connecting two vertices having different colour.

Then \( \text{Per}(A) > 0 \) means that \( A \) has a subgraph which contains \( n \) red and \( m \) blue vertices and \( n \) disjoint edges, i.e. \( n \) edges which have no common endpoint.
2.3 Asymptotic Representations

We often want to find the approximate value of a quantity, instead of an exact value, in order to compare one number to another.

A very convenient notation for dealing with approximations are $\mathcal{O}$, $\omega$ and $\sim$ notation.

Suppose that $f(n)$ and $g(n)$ are two functions of $n$. Then

(i) $f = \mathcal{O}(g)$ means that $|f(n)| < Ag(n)$

where $A$ is independent of $n$ for all values of $n$ in question.

(ii) $f = \omega(g)$ means that $\frac{f(n)}{g(n)} \to 0$ when $n \to \infty$.

(iii) $f \sim g$ means that $\frac{f}{g} \to 1$.

In particular $f = \mathcal{O}(1)$ means that $f$ is bounded and $f = \omega(1)$ means that $f \to 0$.

Thus $n = \mathcal{O}(n^2)$, $100n^2 + 1000n = \mathcal{O}(n^2)$,

$n = \omega(n^2)$, $100n^2 + 1000n = \omega(n^2)$,

$100n^2 + 100n \sim 100n^2$,

and $\frac{a_0n^p + a_1n^{p-1} + \cdots + a_p}{b_0n^q + b_1n^{q-1} + \cdots + b_q} \sim \frac{a_0}{b_0}n^{p-q}$

if $a_0 \neq 0$, $b_0 \neq 0$.

So far we have defined (e.g.) $f(n) = \mathcal{O}(1)$ or $f(n) = \omega(n)$, but not $\mathcal{O}(1)$ or $\omega(n)$ in isolation. We can however make our definitions more elastic. We may agree that $O(g)$ or $\omega(g)$ denotes an unspecified $f$ such that $f = \mathcal{O}(g)$ or $f = \omega(g)$.

We can then write, for example,

$\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1) = \omega(n)$
meaning thereby if \( f = O(1) \) and \( g = O(1) \) then \( f + g = O(1) \).

It is to be observed that the relation "=", asserted between \( O \) or \( o \) symbols, is not usually symmetrical. Thus \( o(1) = O(1) \) is always true; but \( O(1) = o(1) \) is usually false. We may also observe that \( f \sim g \) is equivalent to \( f = g + o(g) \) or to \( f = g(1 + o(1)) \).

In these circumstances we say that \( f \) and \( g \) are asymptotically equivalent, or that \( f \) is asymptotic to \( g \).

The \( O, o \) and \( \sim \) notations are also frequently used with functions of a real variable \( x \).

(i) \( f(x) = O(g(x)) \) means that \( |f(x)| < A g(x) \)
where \( A \) is independent of \( x \) for all values of \( x \) in question.

(ii) \( f(x) = o(g(x)) \) means that \( \frac{f(x)}{g(x)} \to 0 \).

(iii) \( f(x) \sim g(x) \) means that \( \frac{f(x)}{g(x)} \to 1 \).

Thus \( 10x = O(x) \), \( \sin x = O(1) \), \( x = O(x^2) \)
\( x = o(x^2) \), \( \sin x = o(x) \), \( x + 1 \sim x \),
when \( x \to \infty \) and
\( x^2 = O(x) \), \( x^2 = o(x) \), \( \sin x \sim x \), \( 1 + x \sim 1 \),
when \( x \to 0 \).
\[ e^x = 1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{m!} x^m + O(x^{m+1}), \quad |x| \leq r \]
any fixed \( r \).
\[ \log(1 + x) = x - \frac{1}{2} x^2 + \cdots + \frac{(-1)^{m+1}}{m} x^m + O(x^{m+1}), \quad |x| \leq r \]
any fixed \( r < 1 \).

Let us give one simple example of the concepts we have introduced.
Consider the quantity \( n(\sqrt{n} - 1) \).
\[ n(\sqrt[n]{n} - 1) = n(e^{\frac{\log n}{n}} - 1) \]
\[ = n(1 + \frac{\log n}{n} + O((\frac{\log n}{n})^2) - 1) \]
\[ = \log n + O((\frac{\log n}{n})^2) \]
\[ = \log n + o(1). \]

So we find that \( n(\sqrt[n]{n} - 1) \) is approximately equal to \( \log n \); the difference is \( O((\log n)^2/n) \) which approaches to zero as \( n \) approaches to infinity.

2.4 Matching problem

Let \( \text{Inj}(R, C) \) be the set of all \( 1 \rightarrow 1 \) maps from \( R = \{1, 2, \ldots, n\} \) into \( C = \{1, 2, \ldots, n\} \), \( n \leq m \). For given \( 1 \rightarrow 1 \) map \( f \in \text{Inj}(R, C) \), what is the total number of \( 1 \rightarrow 1 \) maps which have \( r \) coincidences with given \( f \)?

This number divided by \( \text{P}^s_m \) is denoted by \( P^s_m(r) \). That is, \( P^s_m(r) \) is the probability that a randomly chosen \( 1 \rightarrow 1 \) map has exactly \( r \) coincidences with a given \( 1 \rightarrow 1 \) map. It is known [1] that

\[ P^s_m(r) = \frac{1}{r!} \frac{n!}{(n-r)!} \frac{(m-r)!}{m!} P^{s-r}_{m-r}(0) \]

and

\[ P^s_m(r) \sim \frac{1}{r!} \left( \frac{n}{m} \right)^r e^{-\frac{n}{m}} \]

We can restate the above problem into the well known matching problem as follows. Two equivalent decks of cards are well shuffled and matched against each other.

With no loss of generality we can assume the cards in the first deck are arranged in the order \( 1, 2, \ldots, m \). The \( n \) cards in the second deck
are then matched against the positions determined by the first deck. A match occurs at position \(i\) if and only if the \(i\)-th card drawn from the second deck is card number \(i\).

Let \(X_{nm}\) be the number of matches. Then

\[
P( X_{nm} = r ) = P^n_m(r) = \frac{1}{r!} \left( \frac{n}{m} \right)^r e^{-\frac{n}{m}}
\]

In particular when

\[
n = \alpha m, \quad 0 < \alpha \leq 1
\]

the limiting distribution of \(X_{nm}\) is the Poisson distribution with mean \(\alpha\).
CHAPTER 3. MAIN RESULTS

3.1. The first and second moments of permanent of random 
\((0, 1)\)-matrices.

In this dissertation, the first and second moment of permanent of 
\(n \times m\) \((0, 1)\)-matrices is derived and it is shown that almost all matrices 
have permanents asymptotics to this average.

Let \(M(n, m, N)\) be the class of all \(n \times m\) \((0, 1)\)-matrices with a given 
number \(N\) of ones.

Then the total number of matrices \(\omega \in M(n, m, N), \# M(n, m, N)\), is 
given by \({n^m \choose N}\).

Let \(P\) be a uniform distribution on \(M(n, m, N), i.e.,\)

\[ P(\omega) = \frac{1}{{n^m \choose N}} \]

Throughout this paper we always assume that \(n \leq m\).

A permanent of \(n \times m\) matrix \(\omega = (\omega_{ij})\) is defined by

\[ \text{Per}(\omega) = \sum_{\sigma \in \text{Inj}(R, C)} \omega_{\sigma(1)} \omega_{\sigma(2)} \cdots \omega_{\sigma(n)} \]

where \(R = \{1, 2, \cdots, n\}, C = \{1, 2, \cdots, m\}\).

Hence \(\text{Per}(\omega)\) consists of \(mP\) summands.

If \(m = n\), then the terms in \(\text{Per}(\omega)\) are, apart from the sign, just the 
terms in the expansion of \(\text{det}(\omega)\).

\(\text{Per}(\omega)\) can be considered as a random variable on \(M(n, m, N)\).

First we compute the first moment and second moment of \(\text{Per}(\omega)\).
Since
\[
\text{Per}(\omega) = \sum_{\sigma \in \text{Inj}(R, C)} \omega_{1\sigma(1)} \omega_{2\sigma(2)} \cdots \omega_{n\sigma(n)}
\]
\[= \sum_{\sigma} \epsilon_{\sigma}
\]

Here \(\epsilon_{\sigma} = 1\) iff \(\omega_{1\sigma(1)} = \omega_{2\sigma(2)} = \cdots = \omega_{n\sigma(n)} = 1\).

\[
E(\text{Per}(\omega)) = \sum_{\sigma} E(\epsilon_{\sigma}) = n P_n \cdot E(\epsilon_{\sigma_1})
\]

where \(\sigma_1(i) = i\).

On the other hand,

\[
E(\epsilon_{\sigma_1}) = P(\omega_{1\sigma(1)} = 1, \omega_{2\sigma(2)} = 1, \ldots, \omega_{n\sigma(n)} = 1)
\]

\[
= \frac{\binom{nm-n}{N-n}}{\binom{nm}{N}} \frac{(nm-n)!}{(nm)!} \frac{(nm-N)!}{(n m - N)! N!}
\]

\[
= \frac{N!}{(n- n)!} \frac{(nm)!}{(nm-n)!}
\]

\[
= \frac{n P_n}{n m P_n}
\]

Hence we obtain

\[
E(\text{Per}(\omega)) = n P_n \cdot \frac{n P_n}{n m P_n}.
\]

We now turn to the evaluation of the second moment.

The squared permanent of a randomly chosen matrix \(\omega\) is given by
\(( \text{Per}(\omega))^2 = (\text{Per}(\omega))(\text{Per}(\omega)) \)

\[ = \sum_{\sigma'} \varepsilon_{\sigma'} \sum_{\sigma''} \varepsilon_{\sigma''} \]

\[ = \sum_{\sigma' \sigma''} \varepsilon_{\sigma'} \varepsilon_{\sigma''} \]

To compute

\[ E(\varepsilon_{\sigma'}, \varepsilon_{\sigma''}) = P(\omega_{1\sigma'(1)} = \omega_{2\sigma'(2)} = \cdots = \omega_{n\sigma'(n)} = 1, \omega_{1\sigma''(1)} = \omega_{2\sigma''(2)} = \cdots = \omega_{n\sigma''(n)} = 1) \]

let \( r \) be the number of set

\[ \{ i \mid \sigma'(i) = \sigma''(i) \} , \]

\( \varepsilon_{\sigma'} \varepsilon_{\sigma''} = 1 \) iff \( \omega \) has \( 1 \)'s in fixed \( 2(n-r) + r \) entries and has the remaining \( 1 \)'s in any entries.

Thus we have

\[ E(\varepsilon_{\sigma'}, \varepsilon_{\sigma''}) = \frac{\binom{nm - (2n - r)}{N - (2n - r)}}{\binom{nm}{N}} \]

\[ = \frac{(nm - (2n - r))!}{(N - (2n - r))! (nm - N)!} \]

\[ = \frac{(nm)!}{N! (nm - N)!} \]

\[ = \frac{N!}{(nm)!} \frac{(nm)!}{(nm - (2n - r))!} \]

\[ = \frac{1}{\nu \binom{2n-r}{2n-r}} \]

Given 1-1 map \( \sigma' \), the number of 1-1 maps \( \sigma'' \) which have \( r \) coincidences with \( \sigma' \) is equal to

\[ - 14 - \]
\[ mP_n \times P \] (a randomly chosen 1-1 map has exactly \( r \) coincidence with a given 1-1 map)

\[ = mP_n \times P(X_{am} = r) \]

by a result in the section 2.4, where \( P_n^r = P(X_{am} = r) \) is given by

\[
\frac{1}{r!} \frac{n!}{(n-r)!} \frac{(m-r)!}{m!} P_{m-r}^r(0)
\]

which is approximately equal to

\[
\frac{1}{r!} \left( \frac{n}{m} \right)^r e^{-\frac{n}{m}}.
\]

Thus we obtain

\[
E(\text{Per}(\omega))^2 = \sum_{\sigma_1 \sigma_2} E(\epsilon_{\sigma_1} \epsilon_{\sigma_2})
\]

\[
= \frac{mP_n}{mP_n} \sum_{r=0}^{n} \frac{n!}{r! (n-r)!} \frac{(m-r)!}{m!} P_{m-r}^r(0) \frac{nP_{2n-r}}{nP_{2n-r}}
\]

\[
\sim \left( \frac{n}{m} \right)^2 \sum_{r=0}^{n} \frac{nP_{2n-r}}{nP_{2n-r}} \frac{1}{r!} \left( \frac{n}{m} \right)^r e^{-\frac{n}{m}}
\]

**THEOREM 3.1.1**

\[
E(\text{Per}(\omega)) = \frac{nP_n}{mP_n}
\]

\[
E(\text{Per}(\omega))^2 = \frac{mP_n}{mP_n} \sum_{r=0}^{n} \frac{n!}{r! (n-r)!} P_{m-r}^r(0) \frac{nP_{2n-r}}{nP_{2n-r}}
\]

\[
\sim \left( \frac{n}{m} \right)^2 \sum_{r=0}^{n} \frac{nP_{2n-r}}{nP_{2n-r}} \frac{1}{r!} \left( \frac{n}{m} \right)^r e^{-\frac{n}{m}}
\]
3.2 Asymptotic estimate

Lemma 3.2.1 Let $m$ and $k$ approach to $\infty$ such a way that $k^3/m^2 \to 0$.

Then

$$\prod_{j=0}^{k-1} (m-j) = m^k \exp \left( -\frac{k^2}{2m} + o\left(\frac{k}{m^{7/6}}\right)\right)$$

Proof

$$\prod_{j=0}^{k-1} (m-j) = m^k \prod_{j=0}^{k-1} (1 - \frac{j}{m})$$

$$= m^k \exp \left[ \sum_{j=0}^{k-1} \log \left( 1 - \frac{j}{m} \right) \right]$$

$$= m^k \exp \left[ \sum_{j=0}^{k-1} \left( -\frac{j}{m} + \frac{1}{2} \left( \frac{j}{m} \right)^2 + \cdots \right) \right]$$

$$= m^k \exp \left[ -\frac{1}{2m} \sum_{j=0}^{k-1} j + O\left( \frac{k^3}{m^2} \right) \right]$$

$$= m^k \exp \left[ -\frac{k^2}{2m} + O\left( \frac{k}{m} \right) + O\left( \frac{k^3}{m^2} \right) \right]$$

$$= m^k \exp \left[ -\frac{k^2}{2m} + o\left( \frac{k}{m^{7/6}} \right) \right].$$

Here we used Taylor series expansion of $\log \left( 1 - \frac{j}{m} \right)$ and $\frac{k}{m} = o\left( \frac{k}{m^{7/6}} \right)$. 

and $k^3/m^2 = \left( \frac{k}{m^{7/6}} \right)^3 = o\left( \frac{k}{m^{7/6}} \right)$.

Lemma 3.2.2

$$\sum_{i \geq L} \frac{x^i}{i!} \leq e^x \frac{x^L}{L!}$$

Proof

$$\sum_{i \geq L} \frac{x^i}{i!} = \frac{x^L}{L!} + \frac{x^{L+1}}{(L+1)!} + \frac{x^{L+2}}{(L+2)!} + \cdots$$

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\[ \frac{x^L}{L!} \left( 1 + \frac{x}{L+1} + \frac{x^2}{(L+1)(L+2)} + \cdots \right) \]
\[ \leq \frac{x^L}{L!} \left( 1 + \frac{x}{2!} + \cdots \right) \]
\[ = e^x \frac{x^L}{L!} . \]

**Theorem 3.2.3** Let \( m = \beta n \), \( \beta \geq 1 \) and \( \frac{N}{m^{3/2}} \to \infty \), then

\[ E(\text{Per}(\omega)) \sim_{\beta^n P_n} \left( \frac{N}{\beta n^2} \right)^n \exp\left[ -\frac{1}{2} \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \]

and

\[ E(\text{Per}(\omega)^3) \sim_{\beta^n P_n^2} \left( \frac{N}{\beta n^2} \right)^2 \exp\left[ -\left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \]

**Proof** Applying Lemma 3.2.1, we see that

\[ \beta^n P_n = N^n \exp\left[ -\frac{n^2}{2N} + o\left( \frac{n}{N^{3/2}} \right) \right] \]

\[ \beta^n P_n^2 = \left( \beta n^2 \right)^n \exp\left[ -\frac{n^2}{2\beta n^2} + o\left( \frac{n}{(n^2)^{3/4}} \right) \right] \]

Therefore

\[ \frac{\beta^n P_n}{\beta^n P_n^2} \sim \left( \frac{N}{\beta n^2} \right) \exp\left[ -\frac{n^2}{2} \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] \]

\[ = \left( \frac{N}{\beta n^2} \right)^2 \exp\left[ -\frac{1}{2} \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \]

Thus

\[ E(\text{Per}(\omega)) = \beta^n P_n \frac{\beta^n P_n}{\beta^n P_n^2} \]

\[ \sim_{\beta^n P_n} \left( \frac{N}{\beta n^2} \right)^n \exp\left[ -\frac{1}{2} \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] . \]

We now turn to find the asymptotic expression of the second moment of \( \text{Per}(\omega) \).
Applying the lemma 3.2.1 we obtain

\[ n P_{2n-r} = N^{2n-r} \exp \left\{ - \frac{(2n-r)^2}{2n} + o \left( \frac{n}{N^{\frac{1}{2}}} \right) \right\} \]

\[ \beta n^2 P_{2n-r} = (\beta n^2)^{2n-r} \exp \left\{ \frac{(2n-r)^2}{2 \beta n^2} + o \left( \frac{n}{(\beta n^2)^{\frac{1}{2}}} \right) \right\} \]

Therefore we have

\[ \frac{n P_{2n-r}}{\beta n^2 P_{2n-r}} \sim \left( \frac{N}{\beta n^2} \right)^{2n-r} \exp \left\{ - \frac{(2n-r)^2}{2} \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right\} \]

From theorem 3.1.1.,

\[ E \left( \text{Per} (\omega)^2 \right) \sim (\beta n P_n)^2 \sum_{r=0}^{\infty} \left( \frac{N}{\beta n^2} \right)^{2n-r} \exp \left\{ - \frac{(2n-r)^2}{2} \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right\} \frac{1}{r!} \left( \frac{1}{\beta} \right)^r e^{-\frac{1}{\beta}} \]

\[ = (\beta n P_n)^2 S_1 + (\beta n P_n)^2 S_2 , \]

where

\[ S_1 = \sum_{r=0}^{\infty} \left( \frac{N}{\beta n^2} \right)^{2n-r} \exp \left\{ - \frac{(2n-r)^2}{2} \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right\} \frac{1}{r!} \left( \frac{1}{\beta} \right)^r e^{-\frac{1}{\beta}} , \]

\[ S_2 = \sum_{r=k+1}^{\infty} \left( \frac{N}{\beta n^2} \right)^{2n-r} \exp \left\{ - \frac{(2n-r)^2}{2} \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right\} \frac{1}{r!} \left( \frac{1}{\beta} \right)^r e^{-\frac{1}{\beta}} . \]

We broke up the sum into two parts. Now we are going to estimate

\[ S_1 \text{ and } S_2 . \]

\[ S_2 \leq \sum_{r=k+1}^{\infty} \left( \frac{N}{\beta n^2} \right)^{2n-r} \frac{1}{r!} \left( \frac{1}{\beta} \right)^r \]

\[ = \left( \frac{N}{\beta n^2} \right)^{2n} \sum_{r=k+1}^{\infty} \frac{\beta n^2}{N} \cdot \frac{1}{r!} \left( \frac{1}{\beta} \right)^r \]

\[ \leq \left( \frac{N}{\beta n^2} \right)^{2n} \frac{n^2 / N}{k!} \]

\[ \leq \left( \frac{N}{\beta n^2} \right)^{2n} e^\frac{n^2}{k!} \left( \frac{n^2 / N}{k!} \right)^4 . \]
In the last inequality, we used Lemma 3.2.2.

On the other hand

\[ \frac{n^2}{N} = \frac{n^{3/2}}{N} n^{1/2} = o(n^{1/2}) \]

\[ \frac{kN}{3n^2} \geq C \frac{n^{5/6}}{n^2} N = C \frac{N}{n^{1/6}} = C \frac{N}{n^{5/6}} n^{1/6} \geq C' n^{1/6} . \]

Therefore we have

\[ e^{n^{2/3}} \frac{(n^2/N)^k}{k!} \]

\[ \leq e^{n^{2/3}} \frac{(n^2/N)^k}{(k/3)^k} \]

\[ = \exp \left[ \frac{n^2}{N} - k \log \left( \frac{kN}{3n^2} \right) \right] \]

\[ \leq \exp \left[ C_1 n^{1/2} - n^{5/6} \log (C' n^{1/6}) \right] \]

\[ \leq \exp \left[ -bn^{5/6} \right] , \]

for sufficiently large \( n \).

Thus

\[ S_2 \leq \left( \frac{N}{n^2} \right)^{2n} \exp \left[ -bn^{5/6} \right] \]

for sufficiently large \( n \).

Now we will estimate \( S_1 \).

\[ S_1 = \sum_{r=0}^{2n} \left( \frac{n^{5/6}}{\beta n^2} \right)^{2n-r} \exp \left[ -\frac{1}{2} (2n-r)^2 (\frac{1}{N} - \frac{1}{\beta n^2}) \right] \cdot \frac{1}{r!} \left( \frac{1}{\beta} \right)^r e^{-\frac{1}{\beta}} . \]

Since

\[ \frac{1}{2} (2n-r)^2 \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \]
\[
\frac{1}{2} \left( 4n^2 - 4nr + r^2 \right) \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \\
= 2n^2 \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) - 2nr \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) + \frac{1}{2} r^2 \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \\
= 2n^2 \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) - 2nr \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) + O \left( \frac{n}{N} \right)^{3/4}
\]

and

\[
\frac{n^{3/4}}{N} = \frac{n^{3/4}}{N} \frac{1}{n^{1/4}} \to 0 \quad \text{as} \quad n \to \infty
\]

Thus

\[
\exp \left[ \frac{1}{2} \left( 2n - r \right)^2 \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] \\
\sim \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) + 2n r \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] \\
= \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \exp \left[ 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right]'
\]

Therefore we have

\[
S_1 \sim \sum_{r=0}^{\infty} \left( \frac{N}{\beta n^2} \right)^{2n-r} \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \cdot \exp \left[ 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] \frac{1}{r!} \left( \frac{1}{\beta} \right)^r e^{-\frac{1}{\beta}} \\
= e^{-\frac{1}{\beta}} \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] \\
\cdot \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{1}{\beta} \right)^r \left( \frac{\beta n^2}{N} \right)^r \exp \left[ 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right]'
\]

Now \[ \sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x \left( 1 + O \left( \frac{x^{k+1}}{(k+1)!} \right) \right) \]

Let \[ x = \frac{n^2}{N} \exp \left[ 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] . \]
Then we can show that
\[
\frac{x^{k+1}}{(k + 1)!} \rightarrow 0 \text{ as } N, n \rightarrow \infty.
\]

Thus we have
\[
S_1 \sim e^{\frac{-1}{\beta}} \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right] e^z
\]
\[
= e^{\frac{-1}{\beta}} \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) + \frac{n^2}{N} \right] \exp \left( 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right)
\]

Since
\[
\frac{n^2}{N} \exp \left[ 2n \left( \frac{1}{N} - \frac{1}{\beta n^2} \right) \right] = \frac{n^2}{N} \left( 1 + O \left( \frac{n}{N} \right) \right)
\]
\[
= \frac{n^2}{N} + O \left( \frac{n^2}{N^2} \right)
\]
\[
= \frac{n^2}{N} + \mathcal{O} \left( 1 \right).
\]

Thus
\[
S_1 \sim \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -\frac{1}{\beta} - 2 \left( \frac{n^2}{N} - \frac{1}{\beta} \right) + \frac{n^2}{N} \right]
\]
\[
= \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ \frac{1}{\beta} - \frac{n^2}{N} \right]
\]
\[
= \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -\left( \frac{n^2}{N} - \frac{1}{\beta} \right) \right]
\]

Since \( \frac{n^2}{N} = \mathcal{O} \left( n^{\frac{1}{2}} \right) \), it follows that
\[
S_1 \leq \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left( -dn^{\frac{1}{2}} \right).
\]

On the other hand
\[
S_2 \leq \left( \frac{N}{\beta n^2} \right)^{2n} \exp \left[ -bn^{\frac{5}{6}} \right],
\]

--- 21 ---
Therefore
\[
\frac{S_2}{S_1} \leq \frac{(N/\beta n^2)^{2n} \exp(-bn^{5/4})}{(N/\beta n^2)^{2n} \exp(-dn^{1/2})}
= \exp(-bn^{5/4} + dn^{1/2}) \rightarrow 0
\]

Hence we obtain
\[
E\left((\operatorname{Per}(\omega))^2\right) = \left(\beta_n P_n\right)^2 (S_1 + S_2)
\sim \left(\beta_n P_n\right)^2 S_1
\sim \left(\beta_n P_n\right)^2 \left(\frac{N}{\beta n^2}\right)^{2n} \exp\left[-\left(\frac{n^2}{N} - \frac{1}{\beta}\right)\right]
\]

This concludes the proof of theorem.

**COROLLARY 3.2.4.** Let $N$ and $n$ approach to infinity such a way that
\[
\frac{N}{n^{1/2}} \rightarrow \infty.
\]
Then
\[
\frac{\operatorname{Per}(\omega)}{E(\operatorname{Per}(\omega))} \rightarrow 1 \quad \text{in probability}.
\]

This corollary means that almost all matrices in $M(n, m, N)$ have permanent asymptotic to $E(\operatorname{Per}(\omega))$.

**PROOF** From theorem 3.2.3, we have that
\[
E\left((\operatorname{Per}(\omega))^2\right) \sim \left(E(\operatorname{Per}(\omega))\right)^2.
\]

Hence
\[
\operatorname{Var}(\operatorname{Per}(\omega)) = E\left((\operatorname{Per}(\omega))^2\right) - \left(E(\operatorname{Per}(\omega))\right)^2
\]
\[
= E^2(\operatorname{Per}(\omega)) \left[\frac{E(\operatorname{Per}(\omega))}{E^2(\operatorname{Per}(\omega))} - 1\right]
\]

\[\text{--- 22 ---}\]
\[ = 0 \left( E^2(\text{Per}(\omega)) \right). \]

It follows from the Chebyshev's inequality that

\[ P \{ \omega : \left| \frac{\text{Per}(\omega)}{E(\text{Per}(\omega))} - 1 \right| \geq \varepsilon \} \]

\[ = P \{ \omega : |\text{Per}(\omega) - E(\text{Per}(\omega))| \geq \varepsilon E(\text{Per}(\omega)) \} \approx \frac{\text{Var}(\text{Per}(\omega))}{\varepsilon^2 E^2(\text{Per}(\omega))} \to 0 \]

as \( n \to \infty \).

### 3.3 Asymptotic property of matrix whose permanent is 0

Let \( A_{nm} = \{ \omega \in M(n, m, N) | \text{Per}(\omega) = 0 \} \)

\( B_{nm} = \{ \omega \in M(n, m, N) | \omega \) has at least one zero row or at least \( m-n+1 \) zero columns \}

On the other hand, by Frobenius-König theorem

\( \text{Per}(\omega) = 0 \) if and only if \( \omega \) contains a \( r \times s \) zero submatrix such that \( r + s = m + 1 \).

In particular \( \text{Per}(\omega) = 0 \) if at least one row is zero \((r=1, s=m)\) or if at least \( m-n+1 \) columns are zeros \((r=n, s=m-n+1)\).

Thus \( B_{nm} \subset A_{nm} \).

**Theorem 3.3.1** If \( N = \frac{1}{2}nm \), then \( P(B_{nm} | A_{nm}) \to 1 \) as \( m \gg n \to \infty \).

**Comment.** Since \( B_{nm} \subset A_{nm} \),

\[ P(B_{nm} | A_{nm}) = \frac{P(B_{nm})}{P(A_{nm})} \]

Thus

\[ P(A_{nm}) \sim P(B_{nm}). \]
One can state this result somewhat vaguely in the following way; if the permanent of a random matrix with element 0 and 1 is equal to 0, then under \( N = \frac{1}{2} n m \) this in most cases is due to the presence of a 0-row or \( m-n+1 \) 0-columns.

**Proof**

\[
 P(B_{nm}|A_{nm}) = \frac{P(B_{nm})}{P(A_{nm})} = 1 - \frac{P(B_{nm}^c \cap A_{nm})}{P(A_{nm})}.
\]

Now we assert that

\[
 B_{nm}^c \cap A_{nm} \subseteq \bigcup_{r=2}^{m-1} B_r, n-s+1
\]

where \( B_r = \{\omega : \omega \text{ has a } r \times s \text{ zero submatrix} \} \),

\[ r + s = m + 1. \]

If \( \omega \in B_{nm}^c \cap A_{nm} \) then \( \omega \in A_{nm} \) and so \( \omega \) has \( r \times s \) zero submatrix such that \( r + s = m + 1 \). and \( \omega \in B_{nm}^c \) and so \( \omega \) dose not have zero row and has at most \( m-n \) zero columns.

Thus \( r \) can not be 1 and \( n \).

Therefore we have that

\[
 P(B_{nm}^c \cap A_{nm}) \leq \sum_{r=2}^{m-1} P(B_r, m-r+1).
\]

From

\[
 P(B_{nm}) = \binom{n}{r} \binom{m}{s} \frac{nm - rs}{N} \frac{N}{nm} \frac{nm}{N},
\]

it follows that

\[
 P(B_{nm}^c \cap A_{nm})
\]
\begin{align*}
\leq & \sum_{r=2}^{m-1} \binom{n}{r} \binom{m}{m-r+1} \frac{N}{nm} \\
\leq & \sum_{r=2}^{m-1} \binom{m}{r} \binom{m}{m-r+1} \frac{nm-r(m-r+1)}{nm} \\
\leq & \binom{m}{2} \binom{m}{m-1} \frac{nm-2(m-1)}{N} \binom{m}{m-1} \frac{nm-2(m-1)}{nm} + \binom{m}{2} \binom{m}{m} \frac{nm-2(m-1)}{nm} \\
\quad + \sum_{r=3}^{m-2} \binom{m}{r} \binom{m}{m-r+1} \frac{N}{nm} \\
\leq & 2 \binom{m}{1} \binom{m}{2} \frac{nm-2(m-1)}{nm} \\
\quad + \sum_{r=3}^{m-2} \binom{m}{r} \binom{m}{m-r+1} \frac{nm-3(m-2)}{nm} \\
\end{align*}

Here we used the inequality 
\[ r(m-r+1) \geq 3(m-2) \]
for \( r = 3, \ldots, m-2 \).

Hence we obtain that

\[ P(B_{nm} \cap A_{nm}) \leq 2 \binom{m}{1} \binom{m}{2} \frac{nm-2(m-1)}{nm} + \frac{2m}{m+1} \frac{nm-2(m-2)}{nm} \]
Since
\[
2 \binom{m}{1} \binom{m}{2} \frac{(nm - 2(m - 1))}{N} \binom{nm}{N}
= 2m \frac{m(m - 1)}{2!} \frac{\sum_{m=2}^{(m-1)} \frac{P_n}{N!}}{\sum_{m=0}^n \frac{P_n}{N!}} \\
\leq m^3 \frac{(nm - 2(m - 1))(nm - 1 - 2(m - 1)) \cdots (nm - N + 1 - 2(m - 1))}{(nm)(nm - 1) \cdots (nm - N + 1)}
\]
\[
= m^3 \left( 1 - \frac{2(m - 1)}{nm} \right) \left( 1 - \frac{2(m - 1)}{nm - 1} \right) \cdots \left( 1 - \frac{2(m - 1)}{nm - N + 1} \right),
\]
\[
(2m) \frac{(nm - 3(m - 2))}{N} \frac{\sum_{m=3}^{(m-2)} \frac{P_n}{N!}}{\sum_{m=0}^n \frac{P_n}{N!}} \\
\leq 2^{2m} \frac{(nm - 3(m - 2))(nm - 1 - 3(m - 2)) \cdots (nm - N + 1 - 3(m - 2))}{(nm)(nm - 1) \cdots (nm - N + 1)}
\]
\[
= 2^{2m} \left( 1 - \frac{3(m - 2)}{nm} \right) \left( 1 - \frac{3(m - 2)}{nm - 1} \right) \cdots \left( 1 - \frac{3(m - 2)}{nm - N + 1} \right).
\]

Thus
\[
P \left( B_{*m} \cap A_{*m} \right)
\leq m^3 \left( 1 - \frac{2(m - 1)}{nm} \right) \left( 1 - \frac{2(m - 1)}{nm - 1} \right) \cdots \left( 1 - \frac{2(m - 1)}{nm - N} \right)
+ 2^{2m} \left( 1 - \frac{3(m - 2)}{nm} \right) \left( 1 - \frac{3(m - 2)}{nm - 1} \right) \cdots \left( 1 - \frac{3(m - 2)}{nm - N} \right)
\]
\[
\leq m^3 \exp \left[ - \frac{2(m-1)}{nm} - \frac{2(m-1)}{nm-1} - \cdots - \frac{2(m-1)}{nm-N} \right] \\
+ 2^m \exp \left[ - \frac{3(m-2)}{nm} - \frac{3(m-2)}{nm-1} - \cdots - \frac{3(m-2)}{nm-N} \right] \\
= m^3 \exp \left[ - 2(m-1) \left( \frac{1}{nm} + \frac{1}{nm-1} + \cdots + \frac{1}{nm-N} \right) \right] \\
+ 2^m \exp \left[ - 3(m-2) \left( \frac{1}{nm} + \frac{1}{nm-1} + \cdots + \frac{1}{nm-N} \right) \right] \\
\leq m^3 \exp \left[ - 2(m-1) \left( \log \frac{nm}{nm-N} \right) \right] + 2^m \exp \left[ - 3(m-2) \left( \log \frac{nm}{nm-N} \right) \right] \\
= m^3 \exp \left[ - 2(m-1) \log 2 \right] + 2^m \exp \left[ - 3(m-2) \log 2 \right] \\
= m^3 \frac{1}{2^{m-1}} + 2^m \frac{1}{2^3(m-2)} \\
= \frac{1}{2^m} \left( m^3 \frac{1}{2^{m-2}} + 64 \right) \\
\]

In the last inequality we used the relation

\[
\frac{1}{nm} + \frac{1}{nm-1} + \cdots + \frac{1}{nm-N} \\
\geq \int_0^N \frac{1}{nm-x} \, dx \\
= \log \frac{nm}{nm-N} 
\]

Thus

\[
P \left( B_{nm} \cap A_{nm} \right) < \frac{C}{2^m} 
\]

for sufficiently large \( m \).

On the other hand,
\[ P(A_{nm}) \leq P(B_{nm}) \leq P\{ \omega : \omega \text{ has at least one zero row} \} \]

\[ \approx \frac{n \binom{nm-m}{N}}{\binom{nm}{N}} - \binom{n}{2} \frac{\binom{nm-2m}{N}}{\binom{nm}{N}} \]

\[ \approx \frac{n \binom{nm-m}{N}}{nm} - \binom{n}{2} \frac{1}{2^m} \]

Hence we have

\[ P(B_{nm}^c | A_{nm}) = \frac{P(B_{nm}^c \cap A_{nm})}{P(A_{nm})} \]

\[ \approx \frac{C}{2^m} \frac{n \binom{nm-m}{N}}{nm} - \binom{n}{2} \frac{1}{2^m} \]

\[ \approx \frac{C}{n2^m \binom{nm-m}{N}} - \binom{n}{2} \frac{1}{2^m} \]

It remains to prove that

\[ n2^m \frac{\binom{nm-m}{N}}{\binom{nm}{N}} \to \infty \text{ as } n \to \infty \]

\[ n2^m \frac{(nm-m)!}{(nm-m-N)!} \frac{(nm-N)!}{(nm)!} \]
\[
\frac{n^2 \sqrt{2 \pi (nm - m)} \left( \frac{nm - m}{e} \right) \sqrt{2 \pi \frac{1}{2} nm} \left( \frac{\frac{1}{2} nm}{e} \right)^{\frac{1}{2} \pi m}}{\sqrt{2 \pi (\frac{1}{2} nm - m)} \left( \frac{\frac{1}{2} nm - m}{e} \right)^{\frac{1}{2} \pi m - m} \left( \frac{nm}{e} \right)^{\pi m}} \\
\sim n^2 \frac{(nm - m)^{\pi m} m^{\pi m - m} (\frac{1}{2} nm)^{\frac{1}{2} \pi m} (nm)^{\pi m}}{(\frac{1}{2} nm - m)^{\frac{1}{2} \pi m - m} (nm)^{\pi m}} \\
\sim n^2 \frac{m^{\pi m - m} (n - 1)^{\pi m - m} \left( \frac{1}{\sqrt{2}} \right)^{\pi m} (nm)^{\pi m}}{m^{\frac{1}{2} \pi m - m} (\frac{1}{2} n - 1)^{\frac{1}{2} \pi m - m} (nm)^{\pi m}} \\
\sim n^2 \frac{(n - 1)^{\pi m - m}}{(\frac{1}{2})^{\frac{1}{2} \pi m} (n - 2)^{\frac{1}{2} \pi m - m}} \\
\sim n^2 \frac{(n - 1)^{\pi m}}{(n - 1)^{\pi m}} \frac{(n - 2)^{\pi m}}{(n - 2)^{\frac{1}{2} \pi m}} \frac{1}{n^{\frac{1}{2} \pi m}} \\
\sim n^2 \frac{(n - 1)^{\pi m}}{(n - 2)^{\frac{1}{2} \pi m} n^{\frac{1}{2} \pi m}} \frac{(n - 2)^{\pi m}}{(n - 1)^{\pi m}} \\
\sim n^2 \frac{(n - 1)^{\frac{1}{2} \pi m}}{n^{\frac{1}{2} \pi m}} \frac{(n - 1)^{\frac{1}{2} \pi m}}{(n - 2)^{\frac{1}{2} \pi m}} \frac{(n - 1)^{-m}}{(n - 2)^{-m}} \\
\sim n^2 \frac{(1 - \frac{1}{n})^{\frac{1}{2} \pi m}}{n^{\frac{1}{2} \pi m}} \left( 1 + \frac{1}{n - 2} \right)^{\frac{1}{2} \pi m - m} \\
\geq n^2 e^{-\frac{1}{2} m} \\
\sim n \left( \frac{2}{\sqrt{e}} \right)^m \to \infty \quad \text{as} \quad n \to \infty.
\]
REFERENCES


LIST OF PUBLICATIONS
On the Expected Value of a Stopped Amart

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停止된 Amart의 期待值에 關於 研究

安位鍾*

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요약 확률 공간을 \((\Omega, F, P)\), sequence 를 \(\{F_n, X_n, n \in \mathbb{N}\}\)라 하자, 단 \(F_n\)는 \(F\)에 있는 빈 텍스트이이고, \(X_n\)의 각각을 \(L'(F_n), n \in \mathbb{N}\)라고 가정한다.

본論文에서는 amart \(\{X_n, n \in \mathbb{N}\}\)에 대하여 sequence \(\{\int \alpha X_n, n \in \mathbb{N}\}\)가 유계라는 것과 각 중 척 \(\nu\)에 대하여 \(\int_{(\nu, \infty)} |X_n|\)가 유한이라는 것은 동등한 것을 밝혔다.

1. Introduction

Let \((\Omega, F, P)\) be a probability space and consider a sequence \(\{F_n, X_n, n \in \mathbb{N}\}\), where the \(F_n\) form a nondecreasing sequence of \(\sigma\)-fields contained in \(F\) and each \(X_n\) is assumed to be in \(L'(F_n), n \in \mathbb{N}\). \(\{X_n, n \in \mathbb{N}\}\) is said to be an amart if the family

\[
\left\{ \int_{\alpha} X_n, \tau \in T \right\}
\]

converges. Fore \(T\) denotes the set of all bounded stopping times \(\tau\). Amarts have been introduced by Edgar and Sucheston \(\cite{3}\) as a generalization of martingales. In this paper we shall prove that for amarts the boundedness of the sequence \(\left\{ \int \alpha X_n, n \in \mathbb{N} \right\}\) and the finiteness \(\int_{(\nu, \infty)} |X_n|\) for each stopping time \(\nu\) are equivalent. The martingale version of this theorem

has been proved by Dubins and Freedman [2]. Following the programme of Edgar and Sucheston in developing the theory of amarts independently of martingale theory we shall give a direct proof to our theorem, though it also may be obtained from the martingale version (Using the Riesz decomposition of an amart into a martingale and an amart converging to zero [3]). Moreover, our proof, which modifies a method of Chow [1], yields the existence of a stopping time \( \nu \) such that \( \int_{(v,\infty]} |X_\tau| = \infty \) even in some cases when the sequence \( \{X_n, n \in \mathbb{N}\} \) has weaker properties than amarts do (corollary 2). We begin with a simple but very useful lemma.

2. Lemma and Theorem

Lemma. Let \( \{X_n, n \in \mathbb{N}\} \) be an amart and \( A \) a set in \( \bigcup_{n=1}^{\infty} F_n \).

Then

\[
\left\{ \int_A X_\tau, \; \tau \in T \right\}
\]

converges.

Proof. Given \( \varepsilon > 0 \) there is, by hypothesis, some \( N \in \mathbb{N} \) such that \( A \in F_N \) and

\[
(*) \quad \left| \int_A X_\tau - \int_A X_\sigma \right| < \varepsilon
\]

holds for all \( \tau, \sigma \in T \) with \( \tau, \sigma > N \). For \( \tau \in T, \; \tau > N \), define

\[
\tau_N(w) = \begin{cases} 
\tau(w) & \text{if } w \in A \\
N & \text{if } w \in \Omega - A
\end{cases}
\]

\( \tau_N \) then is a bounded stopping time and \( \tau_N > N \). The definition of \( \tau_N \) and \((*)\) yield

\[
\left| \int_A X_\tau - \int_A X_\sigma \right| = \left| \int_A X_{\tau_N} - \int_A X_{\sigma_N} \right| < \varepsilon,
\]

which proves the lemma.

Theorem. Let \( \{X_n, n \in \mathbb{N}\} \) be an amart. Then the sequence \( \left\{ \int_A |X_n|, n \in \mathbb{N} \right\} \) is bounded if and only if \( \int_{(v,\infty]} |X_\tau| \) is finite for each stopping time \( \nu \).

proof. Assume first that \( \left\{ \int_A |X_n|, n \in \mathbb{N} \right\} \) is bounded. Then \( \{ |X_n|, n \in \mathbb{N} \} \) is an amart and \( \sup_{\tau} \int_A |X_\tau| \) is finite [3]. Now approximate an arbitrary stopping time \( \nu \) by the truncation \( \nu_n = \nu \wedge n \), which belong to \( T \). On the set \( \{ \nu < \infty \} \) the sequence \( \{ |X_n|, n \in \mathbb{N} \} \) converges to \( |X_\nu| \) and, by Fatou's lemma, one has

\[
- 33 -
\]
\[ \int_{(\alpha,\omega)} |X_\omega| \leq \liminf \int_{\alpha} |X_{\omega}| \leq \sup_T \int_\alpha |X_t| < \infty. \]

Krengel and Suchestor [5] used a similar argument in a more general context (some results of [5] are published without proof in [4]).

Assume now that \( \{ \int_{\alpha} |X_\omega|, \ \omega \in \mathcal{F} \} \) is bounded. We shall show that there exists a stopping time \( \nu \) with \( \int_{(\alpha,\omega)} |X_\omega| = \infty \). To this end we shall construct a decreasing sequence of sets \( C_k \) \( (k=0,1,2,\ldots) \) such that

1. \( (1,0) \) \( C_0 \subseteq F_1 \) and
2. \( (2,k) \) \( \int_{C_k} |X_n^+| \to \infty \), when \( n \) tends to infinity.

Simultaneously, we shall construct a sequence of mutually disjoint sets \( D_k (k=1,2,\ldots) \) and select an ascending subsequence of random variables \( X_n \), belonging to the amart \( (k=1,2,\ldots) \) such that

1. \( (1,k) \) \( C_k \subseteq F_{n_k} \),
2. \( (3,k) \) \( D_k \subseteq F_{n_k} \), and
3. \( (4,k) \) \( \int_{D_k} |X_{n_k}| \geq 1 \).

Start with \( C_0 = \emptyset \). Since \( \{ X_n, n \in \mathbb{N} \} \) is an amart there is some number \( a_0 \) such that

\[ \int_{C_0} X_n^+ - \int_{C_0} X_n^- = \int_{C_0} X_n \to a_0, \]

and by hypothesis one has

\[ \int_{C_0} X_n^+ + \int_{C_0} X_n^- = \int_{C_0} |X_n| \to \infty. \]

Thus \( (1,0) \) and \( (2,0) \) are fulfilled.

Fix now some \( \epsilon > 0 \) and, for general \( k=1,2,\ldots \), assume that \( C_{k-1} \) is given and satisfies \( (1, k-1) \) and \( (2, k-1) \). This will enable us to choose \( X_{n_k} \) and to construct \( C_k \) and \( D_k \) in the manner described above. By \( (1, k-1) \) and the lemma there is some number \( a_{k-1} \) such that

\[ \int_{C_{k-1}} |X_{n_{k-1}}-a_{k-1}|. \]

For \( n \) sufficiently large this yields

\[ \int_{C_{k-1}} X_n^- + |a_{k-1}| + \epsilon > \int_{C_{k-1}} X_n^+. \]

By \( (2, k-1) \) there is some \( n_k > n_{k-1} \) such that

\[ \int_{C_{k-1}} X_{n_k}^+ > 1 + |a_{k-1}| + \epsilon > 1. \]
Together with (5) this implies

\[ \int_{c_{k-1}} X_{n+} \geq 1. \]

Put \( A_k = C_{k-1} \cap \{ X_{n+} > 0 \} \) and \( B_k = C_{k-1} \cap \{ X_{n+} < 0 \} \).

Then

\[ \int_{A_k} X_{n+} + \int_{B_k} X_{n+} = \int_{C_{k-1}} X_{n+} \]

for all \( n \in \mathbb{N} \). By (2, \( k-1 \)) we can choose \( C_k \in \{ A_k, B_k \} \) such that

\[ \int_{C_k} X_{n+} \to \infty, \quad \text{when } n \text{ tends to infinity.} \]

Evidently \( C_k \in \mathcal{F}_n \), so (1, \( k \)) and (2, \( k \)) hold.

By construction of \( A_k \) and (6) we obtain \( \int_{A_k} \mid X_{n_k} \mid > 1 \).

Similarly, using (7), we get \( \int_{B_k} \mid X_{n_k} \mid > 1 \).

let \( D_k = C_{k-1} \setminus C_k \). Then \( D_k \) equals either \( A_k \) or \( B_k \). In any case \( D_k \) is disjoint from \( D_m \) for \( 1 < m < k \), and

\[ \int_{D_k} \mid X_{n_k} \mid > 1. \]

Thus (3, \( k \)) and (4, \( k \)) are valid.

Finally we define the desired stopped time \( \nu \) by letting

\[ \nu(w) = \begin{cases} n_k, & \text{if } w \in D_k, \\ \infty, & \text{if } w \in \Omega - \sum_{k=1}^{\infty} D_k. \end{cases} \]

The definition of \( \nu \) and (8) yield

\[ \int_{(\omega, \nu = \infty)} \mid X_{n_k} \mid = \sum_{k=1}^{\infty} \int_{(\omega = n_k)} \mid X_{n_k} \mid = \infty. \]

The proof of the theorem is thus complete.

The lemma and the second part of the proof of the theorem have two consequences.

**Corollary 1.** Let \( \{ X_n, n \in \mathbb{N} \} \) be an amart and \( A \) a set in \( \mathcal{F}_n \). Then \( \{1_A X_n, n = k, k+1, \ldots \} \) is also an amart.

**Corollary 2.** Let \( \{ X_n, n \in \mathbb{N} \} \) be a sequence such that \( \int_A X_n \) converges for each \( A \in \mathcal{U}_{n=1}^{\infty} \mathcal{F}_n \). Then if

\[ \int \mid X_n \mid \to \infty, \]

there exists a stopping time \( \nu \) with the property that

\[ \int_{(\omega, \nu = \infty)} \mid X_{n_k} \mid = \infty. \]
REFERENCES

On the Largest Optimal Stopping Time.

Wi Chong Ahn*, Bong Dae Choi** and Jae Kyu Lim***

ABSTRACT

The structure of the largest optimal stopping time in the discrete parameter processes is obtained by using the Doob decomposition of supermartingales.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. We denote \(N = \{0, 1, 2, \ldots\}\) and \(\mathcal{N} = N \cup \{+\infty\}\). Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be an increasing sequence of sub-\(\sigma\)-fields of \(\mathcal{F}\), and \((X_n)_{n \in \mathbb{N}}\) a sequence of integrable random variables such that \(X_n\) is \(\mathcal{F}_n\)-measurable. A function \(T : \Omega \rightarrow \mathcal{N}\) is a stopping time if \(\{T \leq n\} \in \mathcal{F}_n\) for all \(n \in \mathbb{N}\). For a stopping time \(T\), a new random variable \(X_T\) is defined by \(X_T(\omega) = X_{T(\omega)}(\omega)\). An optimal stopping time is a stopping time \(T_0\) such that \(E(X_{T_0}) = \sup_T E(X_T)\), where supremum is taken over the set of all stopping times.

Snell (1953) studied the existence and structure of the smallest optimal stopping time for discrete parameter stochastic processes. Later, the general theory of optimal stopping times was developed by Chow, Robbins, and Siegmund (1971). In this paper we study the structure of the largest optimal stopping time for discrete parameter stochastic processes. The largest optimal stopping time \(T\) means the optimal stopping time \(T\) and \(S \leq T\) for any optimal stopping time \(S\).

\((X_n)_{n \in \mathbb{N}}\) is a supermartingale if \(X_n\) is integrable and \(E(X_n | \mathcal{F}_m) \leq X_m\) for each \(n \leq m\). Note that if \((X_n)_{n \in \mathbb{N}}\) is a supermartingale, then by the optional sampling theorem \(E(X_T | \mathcal{F}_S) \leq X_S\) holds for any stopping times \(S\) and \(T\) with \(S \leq T\).

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The following theorem is the fundamental one to study optimal stopping times.

**Theorem 1.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of integrable random variables. Then there is a minimal supermartingale \((Z_n)_{n \in \mathbb{N}}\) satisfying \(X_n \leq Z_n\) for all \(n\). Furthermore
\[
Z_n = \text{ess sup}_{\delta \geq n} E(X_\delta | \mathcal{F}_n),
\]
\[
Z_n = \max\{X_n, E(Z_{n+1} | \mathcal{F}_n)\},
\]
\[
E(Z_n) = \sup_{\delta \geq n} E(X_\delta),
\]
\[
Z_\infty = X_\infty.
\]

In this case, \((Z_n)_{n \in \mathbb{N}}\) is called the **Snell Envelope** of given \((X_n)_{n \in \mathbb{N}}\).

**Proof.** The proof of this theorem is similar to the proof in the case \((X_n)_{n \in \mathbb{N}}\) with \(E(\sup X_n) < +\infty\) (see Neveu(1975) Proposition VI-1-2).

**Lemma 2.** Let \((Z_n)_{n \in \mathbb{N}}\) be the Snell Envelope of \((X_n)_{n \in \mathbb{N}}\). A stopping time \(T\) is an optimal stopping time if and only if \(X_T = Z_T\) and \((Z_{T\wedge n})_{n \in \mathbb{N}}\) is a martingale.

**Proof.** By optional sampling theorem for the supermartingale \((Z_n)_{n \in \mathbb{N}}\), we have \(E(Z_T | \mathcal{F}_0) \leq Z_0\) and \(E(Z_{T\wedge n} | \mathcal{F}_n) \leq Z_{T\wedge(n-1)}\).

Thus \((Z_{T\wedge n})_{n \in \mathbb{N}}\) is a martingale i.e., \(Z_{T\wedge n} = E(Z_T | \mathcal{F}_n)\) if and only if \(E(Z_T) = E(Z_0)\). On the other hand, we have, for every stopping time \(T\),
\[
X_T \leq Z_T,
\]
\[
E(X_T) \leq E(Z_0) = \sup_{\delta} E(X_\delta).
\]

Thus we obtain that
\[
T\text{ is optimal stopping time iff } E(X_T) = E(Z_0)
\]
\[
\text{iff } E(X_T) = E(Z_T) = E(Z_0)
\]
\[
\text{iff } X_T = Z_T, \ E(Z_T) = E(Z_0)
\]
\[
\text{iff } X_T = Z_T, \ (Z_{T\wedge n})_{n \in \mathbb{N}}\text{ is a martingale.}
\]

We recall the **Doob decomposition** of supermartingale: every integrable supermartingale \((Z_n)_{n \in \mathbb{N}}\) can be written in a unique way as the difference of an integrable martingale \((M_n)_{n \in \mathbb{N}}\) and an increasing process \((B_n)_{n \in \mathbb{N}}\), say \(Z_n = M_n - B_n\).

Further uniform integrability of \((Z_n)_{n \in \mathbb{N}}\) is equivalent to regularity of the martingale \((M_n)_{n \in \mathbb{N}}\) together with the condition \(B_\infty = L'\) (see Neveu(1975) Proposition VIII-1-2).

The following our main result shows the structure of the largest optimal stopping time.

**Theorem 3.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of integrable random variables such that
$E(\sup_{n \in \mathbb{N}^+} X_n^+) < +\infty$. Let $(Z_n)_{n \in \mathbb{N}^+}$ be the Snell Envelope of $(X_n)_{n \in \mathbb{N}^+}$ and $Z_n = M_n - B_n$ the Doob decomposition of supermartingale $(Z_n)_{n \in \mathbb{N}^+}$.

Then the stopping time $T$ defined by

$$T(\omega) = \inf \{ n : B_{n+1}(\omega) > 0 \}$$

is the largest optimal stopping time i.e., $T$ is an optimal stopping time and $S \leq T$ for any optimal stopping time $S$.

**Proof.** We first show that supermartingale $(Z_n)_{n \in \mathbb{N}^+}$ is uniformly integrable. The inequality $E(Z_n | \mathcal{F}_n) \leq Z_n$ implies that $(Z_n)_{n \in \mathbb{N}^+}$ is uniformly integrable.

By our hypothesis we have $Z_n^+ \leq E(\sup_{n \in \mathbb{N}^+} X_n^+ | \mathcal{F}_n)$, which implies uniformly integrability of $(Z_n^+)_{n \in \mathbb{N}^+}$. Thus $(Z_n)_{n \in \mathbb{N}^+}$ is uniformly integrable. Next we will show that $T$ is an optimal stopping time. We know that $B_{n+1} = 0$ on the event $\{ T > n \}$ by the definition of the stopping time $T$. Consequently $Z_{n+1} = M_{n+1}$ on $\{ T > n \}$.

As a result,

$$E(Z_{n+1} | \mathcal{F}_n) 1_{\{ T > n \}} = E(Z_{n+1} 1_{\{ T > n \}} | \mathcal{F}_n)$$

$$= E(M_{n+1} 1_{\{ T > n \}} | \mathcal{F}_n) - E(M_n 1_{\{ T > n \}} | \mathcal{F}_n)$$

Thus we have

$$E(Z_{T \wedge (n+1)} | \mathcal{F}_n) = E(Z_T 1_{\{ T \leq n \}} + Z_{n+1} 1_{\{ T > n \}} | \mathcal{F}_n)$$

$$= Z_T 1_{\{ T \leq n \}} + E(Z_{n+1} | \mathcal{F}_n) 1_{\{ T > n \}}$$

$$= Z_T 1_{\{ T \leq n \}} + Z_n 1_{\{ T > n \}}$$

$$= Z_{T \wedge n}.$$  

The above equation shows that $(Z_{T \wedge n})_{n \in \mathbb{N}^+}$ is an integrable martingale. From the previous remark $(Z_{T \wedge n})_{n \in \mathbb{N}^+}$ is uniformly integrable martingale and thus $(Z_{T \wedge n})_{n \in \mathbb{N}^+}$ is a martingale i.e., $E(Z_T | \mathcal{F}_n) = Z_{T \wedge n}$. Next we claim that $X_T = Z_T$. On the event $\{ T = n \}$ we see that $B_n = 0$ and $B_{n+1} > 0$ by the definition of the stopping time $T$.

Thus we have

$$E(Z_{n+1} | \mathcal{F}_n) 1_{\{ T = n \}} = E(Z_{n+1} 1_{\{ T = n \}} | \mathcal{F}_n)$$

$$= E((M_{n+1} - B_{n+1}) 1_{\{ T = n \}} | \mathcal{F}_n) = E(M_{n+1} 1_{\{ T = n \}} | \mathcal{F}_n) - B_{n+1} 1_{\{ T = n \}}$$

$$= (M_n - B_{n+1}) 1_{\{ T = n \}} < M_n 1_{\{ T = n \}} = Z_n 1_{\{ T = n \}}.$$  

From the equality $Z_n = \max(X_n, E(Z_{n+1} | \mathcal{F}_n))$ and the above inequality, we have $Z_n = X_n$ on $\{ T = n \}$. Clearly we have $Z_n = X_n$ on $\{ T = \infty \}$. Therefore $Z_T = X_T$ is obtained. Thus Lemma 2 shows that $T$ is an optimal stopping time. It remains to show that $T$ is the largest optimal stopping time. Let $S$ be any optimal stopping time.
Then by Lemma 2 we have that $X_s = Z_s$ and $(Z_{s_{\wedge}n})_{n \in \mathbb{N}}$ is a martingale.

Thus for each $n \in \mathbb{N}$

$$E(Z_n) = E(Z_{s_{\wedge}n}) = E(M_{s_{\wedge}n} - B_{s_{\wedge}n})$$

$$= E(M_{s_{\wedge}n}) - E(B_{s_{\wedge}n}).$$

On the other hand, from the uniform integrability of $(Z_s)$ we know that $(M_s)_{s \in \mathbb{R}}$ is a regular martingale. Therefore $E(M_{s_{\wedge}n}) = E(M_0)$.

Thus we conclude that $E(M_0) = E(Z_0) = E(M_{s_{\wedge}n}) = E(B_{s_{\wedge}n}) = E(M_0) - E(B_{s_{\wedge}n})$. This shows that $E(B_{s_{\wedge}n}) = 0$ and hence $B_{s_{\wedge}n} = 0$ because $B_{s_{\wedge}n} \geq 0$. By the definition of the stopping time $T$, we have $S_{\wedge}n \leq T$ for all $n$ and hence we have $S \leq T$.

REFERENCES


ON THE UNIFORM INTEGRABILITY OF CONTINUOUS PARAMETER STOCHASTIC PROCESSES

BY WI CHONG AHN, BONG DAE CHOI AND JAE KYU LIM

J.L. Doob (1975) [6] introduced a notion of optionally separable processes which generalizes both the separable processes and well-measurable processes.

G. Johnson and L.L. Helms [8] showed that right continuous supermartingale \((X_t)_{0 \leq t < \infty}\) is of class (D) if and only if \(\lim E(X_{T_n}) = E(X_\infty)\) for every increasing sequence \((T_n)\) of stopping times converging to \(+\infty\) (also see [10, p.102]). In this paper we will extend G. Johnson and L.L. Helms' result to the optionally separable processes.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and \((\mathcal{F}_t)_{0 \leq t < \infty}\) an increasing right continuous family of sub-\(\sigma\)-algebra. \(\mathcal{F}_0\) includes all of the null sets. A process \((X_t)\) is adapted if \(X_t\) is \(\mathcal{F}_t\)-measurable for each \(t\). Unless some other convention is stated explicitly, process \((X_t)\) means a stochastic process \((X_t)_{0 \leq t < \infty}\) adapted to \((\mathcal{F}_t)\). A function \(T : \Omega \to \mathbb{R}_+ \cup \{+\infty\}\) is a stopping time for \((\mathcal{F}_t)\) iff \(\{T \leq t\} \in \mathcal{F}_t\) for all \(t \in \mathbb{R}_+\). It is known that \(\mathcal{F}_T = \{F \in \mathcal{F} : F \cap \{T \leq t\} \in \mathcal{F}_t\ for all t \in \mathbb{R}_+\}\) is a \(\sigma\)-algebra. In order that \(X_T\) is \(\mathcal{F}_T\)-measurable, it suffices to assume that the process \((X_t)\) is progressive i.e., for all \(t \in \mathbb{R}_+\), the map \([0, t] \times \Omega \to \mathbb{R}\) defined by \((s, \omega) \mapsto X_s(\omega)\) is measurable with respect to \(\mathcal{A} \cap [0, t] \times \mathcal{F}_t\).

J.L. Doob introduced optionally separable processes [6].

Definition [6]. If \((X_t)\) is a process, a sequence \((S_n)\) of finite stopping times is called an optional separability set for \((X_t)\) if for each \(\omega\), the set \((S_n(\omega) : n \in \mathbb{N})\) contains 0 and is dense in \([0, \infty)\) and the graph of the sample function \(t \mapsto X_t(\omega)\) is in the closure of the graph restricted to the countable dense set \((S_n(\omega) : n \in \mathbb{N})\).

Note that the set \((S_n \setminus k : n, k \geq 1)\) is also an optionally separability set whose stopping times are bounded. If the stopping times \(S_n\) can be chosen constant, then the process is separable. J.L. Doob [6] has shown that every well measurable process is (indistinguishable from) an optionally separable process.

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Definition [9]. Let \((X_t)_{0 \leq t < \infty}\) be a progressive process. \((X_t)\) is called regular supermartingale if for any stopping times \(S\) and \(T\) with \(S \leq T\), we have \(E(X_T^-) < \infty\) and \(E(X_T^\mathcal{F}_S) \leq X_S\).

It is known that a right continuous supermartingale is regular supermartingale [10, p. 98].

We begin with a “maximal” lemma which is known for separable (or “right continuous”) supermartingale [7, p. 353].

**Lemma 1.** Let \((X_t)_{t \in [0, \infty)}\) be a progressive, optionally separable regular supermartingale. Then for each \(\lambda > 0\),

(a) \(\lambda P(\inf_{0 \leq s \leq \infty} X_t(\omega) < \lambda) \leq \int_{0 \leq s \leq \infty} \{\inf_{0 \leq t \leq s} X_t(\omega) < \lambda\} X_s\)

(b) \(\lambda P(\sup_{0 \leq s \leq \infty} X_t(\omega) > \lambda) \leq \int_{0 \leq s \leq \infty} \{\sup_{0 \leq t \leq s} X_t \geq \lambda\} X_s - EX_S + EX_0\)

**Proof.** Let \((S_n)\) be an optionaly separability set such that the \(S_n\) are bounded. For fixed \(n\) let \((S_1', S_2', \ldots, S_n')\) be the rearrangement of \((S_1, S_2, \ldots, S_n)\) in increasing order, then \((S_1', S_2', \ldots, S_n')\) are also stopping times (see [1, 6]).

Let \(A_1 = \{X_{S_1'}(\omega) < \lambda\}\)

\[A_k = \{X_{S_k'}(\omega) \geq \lambda, \ 1 \leq j < k, \ X_{S_j'} < \lambda\}, \ 1 \leq k \leq n\]

\[A_\infty = X_{S_\infty'}(\omega) \geq \lambda, \ 1 \leq j \leq n, \ X_\infty < \lambda\]

Then \(A_1, \ldots, A_n, A_\infty\) are disjoint, \(A_k \in \mathcal{F}_{S_k'}, \ A_\infty \in \mathcal{F}_\infty\) and

\[A_1 \cup \cdots \cup A_n \cup A_\infty = \{\min_{j = 1, \ldots, n} X_{S_j'}(\omega) < \lambda\}\]

Using the supermartingale inequality and the fact \(X_{S_j'}(\omega) \leq \lambda\) on \(A_k\), we find that

\[\int \{\min_{j = 1, \ldots, n} X_{S_j'}(\omega) < \lambda\} X_{\infty} = \sum_{k = 1, \ldots, n} \int A_k X_{S_k'} < \lambda \sum_{k = 1, \ldots, n} \int A_k X_{S_k'} < \lambda \sum_{k = 1, \ldots, n} \int A_k < \lambda \int_{\mathcal{F}_{S_k'}} X_{S_k'} < \lambda \sum_{k = 1, \ldots, n} \int A_k \}

As \(n \to +\infty\), \(\{\min_{j = 1, \ldots, n} X_{S_j'}(\omega) < \lambda\}\) increase to

\[\{\inf_{0 \leq s \leq \infty} X_t(\omega) < \lambda\} = \{\inf_{0 \leq s \leq \infty} X_t(\omega) < \lambda\}\]

by the definition of optionally separability set. By taking \(n \to \infty\), we get

\[\int_{0 \leq s \leq \infty} \{\inf_{0 \leq s \leq \infty} X_t(\omega) < \lambda\} X_s \leq \lambda P(\{\inf_{0 \leq s \leq \infty} X_t(\omega) < \lambda\})\]

(b) As in (a) let \(S_1', \ldots, S_n'\) be a rearrangement of \(S_1, S_2, \ldots, S_n\) in increasing order. Let \(S_1' = 0\) and \(S_\infty' = +\infty\). Since \((X_t)\) is a regular supermartingale, \((X_{S_1'}, X_{S_2'}, \ldots, X_{S_n'}, X_{\infty})\) is a supermartingale.
Let \( M_n = \{ \max_{j=1}^{\infty} X_{S_j} > \lambda \} \). Define
\[
\tau(\omega) = \begin{cases} 
\min_{j=1}^{\infty} [S_j : X_{S_j} > \lambda] & \text{if } \omega \in M_n \\
\infty & \text{if } \omega \notin M_n
\end{cases}
\]
Then \( \tau \) is a stopping time for \( (\mathcal{F}_{S_0}, \mathcal{F}_{S_1}, \ldots, \mathcal{F}_{S_n}, \mathcal{F}_{S_\infty}) \). By the optional sampling theorem, we have \( E(X_{S_0}) \geq E(X_\tau) \), so that
\[
E(X_{S_0}) \geq E(X_\tau) = \int_{M_n} X_\tau + \int_{\Omega \setminus M_n} X_\infty
\geq \lambda P(M_n) + E(X_\infty) - \int_{M_n} X_\infty
\]
Thus we have
\[
\lambda P(M_n) \leq \int_{M_n} X_\infty - E(X_\infty) + E(X_0)
\]
Since \( M_n \) increase to \( \{ \sup_{1 \leq j \leq \infty} X_{S_j} > \lambda \} = \{ \sup_{0 < t < \infty} X_t(\omega) > \lambda \} \), by taking limit as \( n \to \infty \), we obtain
\[
\lambda P(\{ \sup_{0 < t < \infty} X_t > \lambda \}) \leq \int_{\{ \sup_{0 < t < \infty} X_t > \lambda \}} X_\infty - E(X_\infty) + E(X_0).
\]

**Theorem 2.** Let \( (X_t)_{0 \leq t \leq \infty} \) be a progressive, optionally separable process such that for any stopping time \( S \geq S \) with \( EZ_T < \infty \). Then there is the smallest regular supermartingale \( (Z_t)_{0 \leq t \leq \infty} \) satisfying \( X_t \leq Z_t \) for all \( t \). Furthermore \( (Z_t)_{0 \leq t \leq \infty} \) is progressive, optionally separable process and \( Z_T = \sup_{S \leq T} E(X_S | \mathcal{F}_S) \) for any stopping time \( T \).

\( (Z_t)_{0 \leq t \leq \infty} \) is called the Snell envelope of \( (X_t)_{0 \leq t \leq \infty} \).

**Proof.** The proof of this theorem is similar to the proof in the case of well-measurable progress [9] and is omitted.

**Theorem 3.** Let \( (X_t)_{t \in [0, \infty]} \) be a progressive, optionally separable process. Assume that \( \sup_{t} |E(X_T)| < \infty \), where supremum is taken over the set of all extended stopping times. If \( \lim E(X_{T_n}) = E(X_\infty) \) for every increasing sequence \( (T_n) \) of extended stopping times converging to \( \infty \), then \( (X_t) \) is of class \( (D) \), i.e., \( (X_T) \) is uniformly integrable over the set of all extended stopping times.

**Remark.** G. Johnson and L. L. Helms [8] showed that a non-negative right continuous supermartingale \( (X_t)_{t \in [0, \infty]} \) is of class \( (D) \) if \( \lim E(X_{T_n}) = E(X_\infty) \) for every increasing sequence \( (T_n) \) of extended stopping times converging to \( \infty \). So Theorem 3 is an extension of this result.

**Proof.** Let \( (Z_t)_{t \in [0, \infty]} \) be the Snell envelope of \( (X_t)_{t \in [0, \infty]} \) in the Theorem 2. Define
\[
R_n = \inf \{ t ; Z_t(\omega) \geq n \} \quad (\inf \phi = \infty)
\]
The \((R_n)\) is an increasing sequence of extended stopping times converging to \(\infty\), since \(\sup Z_t < +\infty\) a.e. by Lemma 1. We will break the remainder of the proof into three steps.

**Step 1.** We will show that \(E(Z_{R_n}) \to E(X_\omega)\). Suppose that \(\lim_{n \to \infty} E(Z_{R_n}) > E(X_\omega) + \varepsilon\) for some \(\varepsilon > 0\). Since \(E(Z_{R_n}) = \sup_{T \geq R_n} E(X_T) > E(X_\omega) + \varepsilon\) for each \(n\), there exists an stopping time \(S_n \geq R_n\) such that \(E(X_{S_n}) > E(X_\omega) + \varepsilon\). It remains to show that we can replace the sequence \((S_n)\) by an increasing sequence. Define

\[
T_n'(\omega) = \min \{ S_k(\omega) \mid S_k(\omega) \geq R_n(\omega), \quad X_{S_k}(\omega) \geq E(X_{S_k} \mid \mathcal{F}_{S_k}) \}
\]

Then \(R_n \leq T_n' \leq S_n\), \(X_{T_n'}(\omega) \geq E(X_{S_n} \mid \mathcal{F}_{T_n'})\) and \(T_n'\) is an extended simple stopping time, because

\[
\{ \omega : S_k(\omega) \geq R_n, \quad X_{S_k}(\omega) \geq E(X_{S_k} \mid \mathcal{F}_{S_k}) \} \in \mathcal{F}_{S_k}. \quad \text{Let } T_n' = \max_{1 \leq j \leq n} T_j.
\]

Then \((T_n')\) is an increasing sequence of extended simple stopping times converging to \(\infty\). Now we will show that \(E(X_{T_n'}) \geq E(X_{T_n'}) \geq E(X_{S_n}) \geq E(X_\omega) + \varepsilon\), which contradicts the hypothesis of theorem. For fixed \(n\), let \(T_{i+1}'' = T_n' \wedge T_i''\), then \(T_{i+1}'' = T_1'' \leq T_2'' \leq \cdots \leq T_{n-1}'' \leq T_n'' = T_n\). We assert that \(E(X_{T_{i+1}''}) \leq E(X_{T_i''})\) for all \(i\), thus we have \(E(X_{T_n'}) \leq E(X_{T_n})\).

\[
E(X_{T_{i+1}'}) = \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} X_{T_i''} dP
\]

\[
\leq \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} E(X_{S_i} \mid \mathcal{F}_{T_i''}) dP
\]

\[
= \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} X_{S_i} dP
\]

\[
= \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} E(X_{S_i} \mid \mathcal{F}_{T_i''}) dP
\]

\[
\leq \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} X_{T_i''} dP
\]

\[
= \int_{T_i'' = T_{i+1}}^T X_{T_{i+1}''} dP + \int_{T_i'' < T_i} X_{T_i''} dP = E(X_{T_{i+1}'})
\]

The first and second inequalities followed from the definition of the stopping time \(T_i''\).

**Step 2.** We will prove that \((Z_t)_{t \in [0, \infty]}\) is of class (D). Let \(T\) be an arbitrary stopping time and \(T'\) stopping time defined by

\[
T'(\omega) = \begin{cases} T(\omega) & \text{if } Z_T(\omega) \geq n \\ +\infty & \text{otherwise} \end{cases}
\]

Then we have \(R_n \leq T'\) and consequently \(E(Z_{R_n}) \geq E(Z_{T'}) \geq E(Z_\omega)\).

By step 1 we obtain that
On the uniform integrability of continuous parameter stochastic processes

$$\int_{\{Z_T > n\}} Z_T + \int_{\{Z_T < n\}} Z_T \to E(Z_\infty) \text{ as } n \to \infty$$

On the other hand, \(\{\sup_{0 \leq t < \infty} Z_t \leq n\} \subseteq \{Z_T \leq n\}\) for any stopping time \(T\). Since 
\(\sup_{0 \leq t < \infty} Z_t \leq +\infty\) a.e. by Lemma 1, \(P(\{\sup_{0 \leq t < \infty} Z_t \leq n\}) \to 1\) as \(n \to \infty\). Therefore
\(P(\{Z_T \leq n\}) \to 1\) as \(n \to \infty\) uniformly on \(T\). Thus

$$\int_{\{Z_T > n\}} Z_T = \int_{\{Z_T > n\}} Z_T^+ \to 0 \text{ as } n \to \infty$$

uniformly in \(T\), which implies \((Z_T^+)_{T \in \mathcal{F}}\) is uniformly integrable where \(\mathcal{F}\) is the set of all extended stopping times. From the relation \(E(Z_\infty | \mathcal{F}_T) \leq Z_T\), we derive uniform integrability of \((Z_T^-)_{T \in \mathcal{F}}\).

Step 3. We will show that \((X_T)_{T \in \mathcal{F}}\) is uniformly integrable. Since \(X_t \leq Z_t\) for all \(t\) and \((Z_T^+)_{T \in \mathcal{F}}\) is uniformly integrable, it follows that \((X_T^+)_{T \in \mathcal{F}}\) is uniformly integrable. In order to prove that \((X_T^-)_{T \in \mathcal{F}}\) is uniformly integrable, consider process \((-X_t)_{t \in [0, \infty]}\) which satisfies all the conditions of the theorem. Using \((-X_T)^+ = X_T^-\) and step 2 we obtain that \((X_T^-)_{T \in \mathcal{F}}\) is uniformly integrable. Thus \((X_T)_{T \in \mathcal{F}}\) is uniformly integrable.

Corollary 4. Let \((X_t)_{t \in \mathbb{R}_+}\) be progressive, optionally separable process with \(\sup |E(X_T)| < +\infty\). If \(E(X_T)_n \to E(X_T)\) for every increasing sequence \((T_n)\) of finite stopping times, which converges to any finite stopping time \(T\), then \((X_{t,T})_{t \in \mathbb{R}_-}\) is of class (D) for any finite stopping time \(T\).

Proof. For any finite stopping time \(T\), let \(Y_t = X_{t,T}\) for \(t \in [0, \infty]\), then \((Y_t)_{t \in [0, \infty]}\) is progressive, optionally separable process. Applying Theorem 3 to the process \((Y_t)\), we obtain that \((Y_t)\) is of class (D).

References


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