Simple Expression of Ergodic Capacity for Rician Fading Channel

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SUMMARY In this letter, a new exact expression of the ergodic channel capacity for a Rician fading channel is provided that is written in terms of exponential integral and incomplete gamma function. Also, a good approximation of the Rician fading channel capacity is derived from the exact expression and its accuracy is numerically verified.

key words: rician channel, ergodic channel capacity

1. Introduction

The ergodic capacity of various fading channels such as Rayleigh and Rician fading channels has been studied for a long time [1], [2]. Some closed-form expressions of the fading channel capacity have been obtained in terms of a higher transcend function of Meijer’s G function [3] or triple summations of the Poisson distribution [2]. Unfortunately, those representations are somewhat complex and their numerical calculation is not efficient. In this letter, a new expression for a Rician fading channel capacity is formulated that is more efficient and stable than the previous ones in numerical calculation. Then, based on our new expression, a simple approximation is derived and its accuracy is verified.

2. Exact Formulation

The ergodic channel capacity of a fading channel can be written as

\[
\langle C \rangle = \int_0^\infty \log_2(1 + \gamma)p_\gamma(\gamma)d\gamma \quad \text{[bits/sec/Hz]},
\]

where \(\gamma\) and \(p_\gamma(\gamma)\) are the received signal-to-noise ratio (SNR) and the probability density function (PDF) of the received SNR, respectively. For a Rician fading channel, the PDF of the received SNR is given [3] by

\[
p_\gamma(\gamma) = \frac{1 + K}{\bar{\gamma}} e^{-K} e^{-\gamma(1+K)/\bar{\gamma}} I_0\left[\sqrt{\frac{4K(1+K)\gamma}{\bar{\gamma}}}\right],
\]

where \(K\) is the Rician factor, \(\bar{\gamma}\) is the corresponding average SNR, and \(I_0(\cdot)\) is the zeroth order modified Bessel function. As a first step, the Bessel function in (1) is expanded in a power series to obtain

\[
\langle C \rangle = \frac{e^{-K}}{\ln 2} \frac{1 + K}{\bar{\gamma}} \sum_{n=0}^\infty \frac{1}{n!} \left[\frac{K(K + 1)}{\bar{\gamma}}\right]^n \int_0^\infty \gamma^n \ln(1 + \gamma)e^{-(1+K)\gamma/\bar{\gamma}}d\gamma.
\]

The integral in (2) can be analytically evaluated as in [4]

\[
\int_0^\infty \gamma^n \ln(1 + \gamma)e^{-(1+K)\gamma/\bar{\gamma}}d\gamma = n!e^\frac{1+K}{\bar{\gamma}} \left(\frac{\bar{\gamma}}{1+K}\right)^{n+1} \sum_{k=0}^n E_{n+1} \left(1 + \frac{K}{\bar{\gamma}}\right),
\]

where \(E_n(\cdot)\) is the exponential integral [5]. Thus, (2) can be written as

\[
\langle C \rangle = \frac{e^{-K}e^\frac{1+K}{\bar{\gamma}}}{\ln 2} \sum_{n=0}^\infty \frac{K^n}{n!} \sum_{k=0}^n E_{n+1} \left(1 + \frac{K}{\bar{\gamma}}\right).
\]

The double summations in (3) can be simplified by rearranging the summation terms from the horizontal to diagonal direction [6] as

\[
\sum_{n=0}^\infty \frac{K^n}{n!} \sum_{k=0}^n E_{n+1} \left(1 + \frac{K}{\bar{\gamma}}\right) = \sum_{n=0}^\infty \frac{K^n}{n!} E_{n+1} \left(1 + \frac{K}{\bar{\gamma}}\right) \left[1 + \frac{K}{n + 1} + \frac{K^2}{(n + 1)(n + 2)} + \cdots\right].
\]

The summation in the bracket of (4) can be written in terms of the confluent hypergeometric function as \(1F_1(1; n + 1; K)\) [5]. After which it can be converted to a known transcend function as \(1F_1(1; n + 1; K) = ne^K K^{\gamma} \gamma(n, K)\), where \(\gamma(\cdot, \cdot)\) is the incomplete gamma function [5]. From above, the ergodic capacity of a Rician fading channel can be expressed in a compact form as

\[
\langle C \rangle = \frac{e^{1+K}}{\ln 2} \sum_{n=0}^\infty E_{n+1} \left(1 + \frac{K}{\bar{\gamma}}\right) P(n, K),
\]

where \(P(a, x) = \gamma(a, x)/\Gamma(a)\) is the regularized gamma function. Using the ratio test [7], it is easy to show that (5) absolutely converges for any values of parameters \(K\) and \(\bar{\gamma}\).
Compared with the known expressions [2], [3], the computation of (5) is much more efficient since the calculation of the Meijer’s G function in [3] is numerically difficult. The formulation in [2] contains triple summations while (5) has only a single summation. Also, the series is very efficient for numerical calculation since both \( P(n, K) \) and \( E_{n+1}(\cdot) \) are monotonically decaying functions of \( n \). Hence, only a few terms are necessary to obtain a sufficient accuracy.

3. Approximate Formulation

When \( K = 0 \), it is easy to show that (5) reduces to the capacity of Rayleigh fading channel [1] since \( P(n, 0) = 0 \) and \( P(0, 0) = 1 \) [5]. On the other extreme, that is, when \( K = \infty \), (5) should be equal to the capacity of an additive white Gaussian noise (AWGN) channel [8]. At this extreme, we can easily characterize the channel capacity if the asymptotic expansion of the capacity is formulated. Asymptotic expansion, \( g(x) \) of \( f(x) \) is defined as \( f(x)/g(x) \sim 1 \) for \( x = \infty \) [6].

When \( \bar{\gamma} = (1 + K)/\gamma \), (5) can be rewritten as

\[
\langle C \rangle \sim \frac{1}{\ln 2} \left\{ e^\alpha E_1(\alpha) \right. \\
+ \int_0^\infty dt e^{-t} \sum_{n=1}^\infty \frac{t^{n-1}}{\Gamma(n)} \left[ \frac{1 - n + 1}{\alpha} + \cdots \right] \right. \right. ,
\]

where \( \alpha = (1 + K)/\bar{\gamma} \). In (6), the summation can be analytically evaluated as

\[
\sum_{n=1}^\infty \frac{t^{n-1}}{\Gamma(n)} \left( \frac{1}{\alpha} - \frac{n + 1}{\alpha^2} + \cdots \right)
= \frac{1}{\alpha^2} \frac{d^2}{dt^2} (te^t) - \frac{1}{\alpha t} \frac{d}{dt} (t^2 e^t) + \frac{1}{\alpha^2 t} \frac{d^2}{dt^2} (t^3 e^t) - \cdots .
\]

Using Leibniz’s differentiation rule [5], (5) can be asymptotically expressed as

\[
\langle C \rangle \sim \frac{1}{\ln 2} \left\{ e^\alpha E_1(\alpha) \right. \\
+ \int_0^\infty dt e^{-t} \sum_{n=1}^\infty \frac{t^{n-1}}{\Gamma(n)} \left[ \frac{1 - n + 1}{\alpha} + \cdots \right] \right. \right. \]
Fig. 2  Relative error between the exact and approximate capacities as a function of $K$ and $\bar{\gamma}$.

Fig. 3  Comparison of the channel capacities computed by (5) and (10) for two Rician fading channels with $K = 0$ dB and $K = 6$ dB.

represents larger relative error. From the figure, the maximum relative error in our range is around 16% and as expected (10) becomes more accurate as $K$ increases. Figure 3 shows the exact and approximate capacities for $K = 0$ dB and $K = 6$ dB. From our examples, the approximate formulation is shown to provide very accurate results for the practical ranges of $K$ and $\bar{\gamma}$.

4. Conclusion

A new closed-form expression for the Rician fading channel capacity has been derived. Our new expression is simpler and more compact than the previous expressions. Based on the exact formulation, an approximation has obtained and verified for a wide range of $K$ and $\bar{\gamma}$. Numerical results show that our approximate expression is very accurate for the practical ranges of $K$ and $\bar{\gamma}$. For the range of $-4$ dB $\leq K \leq 5$ dB, and $-10$ dB $\leq \bar{\gamma} \leq 20$ dB, the discrepancy between the approximate and exact results is less than 16%. The approximate expression is written as a summation of logarithm and a rational function, which is more intuitive than the existing expressions.

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References