Capacity Region of Multiuser Shared Channel with Time-Varying Transmission Power

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Abstract—In this paper, we investigate the capacity region of a multiuser shared channel with time-varying transmission power, whose typical examples are the Forward Packet Data Channel (F-PDCH) of IS-2000 1xEV-DV system and the High Speed Downlink Shared Channel (HS-DSCCH) in 3GPP High Speed Downlink Packet Access (HSDPA) system. The concavity of the throughput yields the property that additional power contributes more to a weaker user having a lower channel gain than to a stronger user. Noting this property, we consider a power-balanced policy which tries to balance the received SNR of each user by serving a weaker user with higher transmission power. Then, we establish the equivalence of the power-balanced policy and the Pareto optimal policy which yields a throughput vector at the boundary of the capacity region. This enables to characterize the capacity region explicitly, and also renders an easy means to maximize the total throughput while meeting each user's requirement.

Index Terms—Capacity region, resource allocation, time-varying transmission power, IS-2000 1xEV-DV, HSDPA.

I. INTRODUCTION

The growing demand for wireless data access requires high-rate data services over wireless channels. However, wireless resources -- transmission power and bandwidth -- are too expensive to increase the data rate simply by consuming more resources. This high-cost nature of wireless resources motivates the development of systems dedicated and optimized to data services, such as CDMA/HDR system [1], along with efficient resource allocation schemes [2]–[7]. Each of those works considered fixed amount of resources dedicated to data traffic users and presented an optimal allocation scheme under its own fairness criterion.

It may be a good solution to high-rate wireless data services to provide a dedicated system, but it has the limitation that it cannot exploit an efficient resource sharing among different types of services: If resources are dedicated to only one type of service, the system cannot accommodate time-varying needs of various different service types (e.g., voice, video streaming, file transfer). In order to overcome this shortcoming, it is necessary to develop a system that supports various types of services in a common wireless access system, thereby enabling an efficient resource sharing among different services.

The above approach necessitates a study on resource allocation problem in the environment of time-varying resources. For example, we consider the IS-2000 1xEV-DV system which was designed to provide high-speed integrated data and voice services while maintaining backward compatibility with the previous CDMA systems [8]: This system first transports voice traffic over a dedicated channel called fundamental channel (FCH) and then allocates all the remaining resources, such as Walsh codes and transmission power, to Forward Packet Data Channel (F-PDCH) dedicated to data traffic users. Among the above two resources, the number of remaining Walsh codes does not change abruptly, as the call duration of voice traffic is sufficiently long when compared with the resource allocation period (i.e., time slot). However, the remaining power may vary frequently in time, as FCHs are power-controlled to compensate for the channel variation of voice traffic users with high mobility. Consequently, in the case of the 1xEV-DV system, it is necessary to devise a downlink resource allocation scheme that can allocate the time-varying power resource to the constituent data users fairly and efficiently. The same necessity arises for the High Speed Downlink Shared Channel (HS-DSCCH) in 3GPP High Speed Downlink Packet Access (HSDPA) as it employs the same channel structure [9], [10].

In this paper, we generalize the above resource allocation problem as a multiuser shared channel problem with time-varying transmission power, formulating it as an optimization problem. Then, we mathematically characterize its capacity region by determining the Pareto optimal policies that yield throughput vectors at the boundary of the capacity region. In solving this problem, we take advantage of the concave property of the transmission rate to derive an equivalent expression of the Pareto optimal policies in terms of the allocated transmission power states. The derived equivalence implies that, in order to achieve the optimality, a higher power state should be allocated to a user with a weaker channel gain. This is because the higher transmission power contributes more to a weaker user but the difference of allocated power states does not affect the transmission rate of a stronger user much due to the concavity of the relation between the transmission power and rate. Owing to this equivalence, it becomes possible to characterize the capacity region explicitly. This also enables an easy achievement of the optimality in solving the given resource allocation problem.
This paper is organized as follows: We first describe the system model in Section II, and formulate the optimization problem to characterize the capacity region in Section III. We present the optimal policy that achieves the boundary of the capacity region in Section IV, and discuss how to maximize the total throughput in Section V. Then, we extend the established results to the fading channels in Section VI and, finally, we show some numerical examples in Section VII.

II. SYSTEM MODEL

We consider a downlink transmission where \( K \) users share common bandwidth \( W \) and transmission power \( P \). We assume that the channel gain between the base station (BS) and a user \( k \), normalized by the noise density, is fixed at the value \( g_k \), with the user numbering made in ascending order of \( g_k \). In addition, we assume that the available transmission power \( P \) is a random variable with \( \Pr(P = P_m) = q_m \) for \( m = 1, 2, \ldots, M \). Transmission power is said at state \( m \) when \( P = P_m \), with the order \( 0 < P_1 < P_2 < \cdots < P_M \).

We consider resource allocation schemes, or policies, that distribute the available transmission power to the constituent users. A policy is said deterministic if the fraction of transmission power allocated to each user at each state, \( \theta_k(m) \), is fixed for \( k = 1, 2, \ldots, K \) and \( m = 1, 2, \ldots, M \). If we employ the conventional matched filter receiver without successive interference cancellation, the maximum available transmission rate is given by

\[
    r_k(m) = W \log \left( 1 + \frac{g_k \theta_k(m) P_m}{1 + g_k (1 - \theta_k(m)) P_m} \right).
\]

A deterministic policy is said feasible if each \( \theta_k(m) \) meets the constraint

\[
    \sum_{k=1}^{K} \theta_k(m) = 1 \text{ and } 0 \leq \theta_k(m) \leq 1.
\]

In this case, the throughput of user \( k \), \( T_k \), yielded by a feasible deterministic policy is given by

\[
    T_k = \sum_{m=1}^{M} q_m r_k(m).
\]

In general, we may employ a feasible deterministic policy for a certain period of time and then change it to another policy at the next time period. More rigorously, let a policy \( \{ \theta_k^{(i)}(m) \} \) be used with a probability \( d_i \), \( i = 1, 2, \ldots, I \), for the number of possible deterministic policies \( I \). Then the overall policy is deterministic if \( d_i \) is 1 for an \( i \) and 0 for all others; and non-deterministic otherwise.

In the case when multiple users share common resources, it is important to determine a policy that can yield a requested throughput vector \( \mathbf{T} \equiv [T_1, T_2, \cdots, T_K] \). A throughput vector \( \mathbf{T} \) is said feasible if there exists a feasible policy that yields a throughput vector \( \tilde{\mathbf{T}} \) such that \( \mathbf{T} \preceq \tilde{\mathbf{T}} \). We define by

\[1\] We say \( x = [x_1, x_2, \ldots, x_K] \succeq y = [y_1, y_2, \ldots, y_K] \) if \( x_k \geq y_k \) for \( k = 1, 2, \ldots, K \). We say \( x \) dominates \( y \) if \( x \succeq y \) and \( x_k > y_k \) for at least one \( k \).

III. PROBLEM FORMULATION

A throughput vector is said to be at the boundary of the capacity region if no component can be increased without crossing over the capacity region. So, any throughput vector at the boundary is Pareto optimal. For efficient resource allocation, it is necessary to determine an optimal throughput vector at the boundary of the capacity region. So we mathematically formulate the capacity region of the system in the following.

We first show that the capacity region is convex, as is proved in the following lemma.

Lemma 1: The boundary of the capacity region is composed of the solutions to the optimization problem (we call it Problem A)

\[
    \max_{\mathbf{T}} \mu \cdot \mathbf{T}
\]

subject to \( \mathbf{T} \in \mathcal{C} \) in (4) for a positive \( \mu \in \mathbb{R}^K_+ \).

Proof: We consider two policies \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) with the throughput vectors \( \mathbf{T}_1, \mathbf{T}_2 \in \mathcal{C} \), respectively. Then, for \( 0 \leq \lambda \leq 1 \), a throughput vector \( \mathbf{T}_3 = \lambda \mathbf{T}_1 + (1 - \lambda) \mathbf{T}_2 \) is contained in \( \mathcal{C} \) since the corresponding policy \( \mathcal{P}_3 \) that is yielded by employing \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) with the probabilities \( \lambda \) and \((1 - \lambda)\) respectively, is feasible. This implies that the capacity region is convex, which proves the lemma.

According to Lemma 1, any point at the boundary of the capacity region is a throughput vector that maximizes \( \mu \cdot \mathbf{T} \) over the capacity region for some \( \mu \in \mathbb{R}^K_+ \). This implies that, for a given Pareto optimal throughput vector, we can find a supporting hyperplane to the capacity region at that point and \( \mu \) is the normal vector of the supporting hyperplane. The normal vector \( \mu \) may be interpreted as the weighting factor that prioritizes the constituent users and, in this sense, it may be said to represent a fairness criterion in resource allocation. By varying \( \mu \), we can get all the points at the boundary of the convex capacity region.

Theorem 1: The maximum of Problem A can be attained only for \( \theta_k^{(i)}(m) = 0 \) or 1.

Proof: By taking the second-order derivative of \( T_k \) in (3), we can get \( \frac{\partial^2 T_k}{\partial \theta_k^{(i)}(m)^2} \geq 0 \) for \( 0 \leq \theta_k^{(i)}(m) \leq 1 \). This implies that the objective \( \mu \cdot \mathbf{T} \) is a convex function of \( \theta_k^{(i)}(m) \). So its maximum value always lies at the boundary, where \( \theta_k^{(i)}(m) \) is either 0 or 1. 

Theorem 1 implies that the optimal policy does not allow power-sharing among users, as it allocates all the transmission power to a single user at a time. This result coincides with
those in [11], [12] even though the considered system models and the problem formulations are different and each result is derived in its own way. By applying Theorem 1 to Problem $A$, we get the following equivalent problem (we call it Problem $B$):

$$\max_{\mathbf{T}} \mathbf{\mu} \cdot \mathbf{T}$$

subject to $T_k = \sum_{i=1}^{I} d_i \sum_{m=1}^{M} q_m R_k(m) \theta_k^{(i)}(m)$,

$$\sum_{i=1}^{I} d_i = 1, \sum_{k=1}^{K} \theta_k^{(i)}(m) = 1,$$

$$\theta_k^{(i)}(m) \in \{0, 1\}, \quad 0 \leq d_i \leq 1,$$  \hspace{1cm} (6)

where

$$R_k(m) = W \log(1 + g_k P_m).$$  \hspace{1cm} (7)

Note that $R_k(m)$ indicates the transmission rate that would yield if all the transmission power $P_m$ were allocated to a single user $k$, as can be derived from (1) by putting $\theta_k(m) = 1$.

Theorem 2: For a given $\mathbf{\mu}$, the deterministic policy

$$\theta_k^{(i)}(m) = \begin{cases} 1, & \text{if } k = k^*(m) = \arg \max_{k=1,2,\ldots,K} \mu_k R_k(m) \\ 0, & \text{otherwise} \end{cases}$$

yields the optimal throughput vector, $\mathbf{T}^*$, of Problem $B$.

Proof: Since $\sum_{i=1}^{I} d_i = 1$ and $\sum_{k=1}^{K} \theta_k^{(i)}(m) = 1$, we get $\sum_{i=1}^{I} d_i \sum_{m=1}^{M} \mu_k R_k(m) \theta_k^{(i)}(m) \leq \max_{m} \mu_k R_k(m)$. So, $\mathbf{\mu} \cdot \mathbf{T} = \sum_{m=1}^{M} q_m \sum_{i=1}^{I} d_i \sum_{k=1}^{K} \mu_k R_k(m) \theta_k^{(i)}(m) \leq \sum_{m=1}^{M} q_m \max_{k} \mu_k R_k(m) = \mathbf{\mu} \cdot \mathbf{T}^*$.

When a tie occurs in selecting $k^*(m)$ in (8) (i.e., when two or more users yield the same maximum value), any user may be selected in random fashion. In this paper, we select the user with the smallest index (i.e., the user with the weakest channel gain) to maintain the deterministic-nature of the policy.

By applying Theorem 2 to Problem $B$, we can further simplify the optimization problem to the following form (we call it Problem $C$):

$$\max_{\mathbf{T}} \mathbf{\mu} \cdot \mathbf{T}$$

subject to $T_k = \sum_{m=1}^{M} q_m R_k(m) \theta_k(m)$,

$$\sum_{k=1}^{K} \theta_k(m) = 1, \quad \theta_k(m) \in \{0, 1\}.$$  \hspace{1cm} (9)

IV. THE OPTIMAL POLICY

The discussions in the previous section have established that the optimal policy allocates transmission power to users in a time-division multiplexing (TDM) manner, serving only one user at a time (i.e., $\theta_k(m) = 0$ or 1). So, we focus on such a TDM policy in investigating the properties of the optimal policy in this section.

We first establish the following lemma that plays a key role in proving the optimality theorems that follow.

Lemma 2: If $0 < \alpha_1 < \alpha_2$, then $\log(1+\alpha_1 x)/\log(1+\alpha_2 x)$ is a monotonically increasing function for $x > 0$.

Proof: We omit the proof as one can easily show that the derivative of the function is positive for $0 < \alpha_1 < \alpha_2$. $

A. Deterministic policy

First, we study the optimal deterministic policy that yields a throughput vector $\mathbf{T}^*$ at the boundary of the capacity region. In particular, we consider the deterministic policy that allocates a higher state to a user having lower channel gain, that is,

$$k^*(1) \geq k^*(2) \geq \cdots \geq k^*(M),$$  \hspace{1cm} (10)

where $k^*(m)$ indicates the index of the user served at state $m$. A deterministic policy that holds the relation in (10) is called power-balanced since the policy tries to balance the received power of each user by serving the user having a lower channel gain with a higher transmission power. As established in the following theorem, the power-balanced policy turns out equivalent to the optimal deterministic policy.

Theorem 3: A deterministic policy is optimal if and only if it is power-balanced.

Proof: First we prove the “only if” part by showing that an optimal deterministic policy in (8) is power-balanced. For notational convenience, we set $a = k^*(m)$ and $b = k^*(m+1)$ for an arbitrary $m$. From (8), we get $\mu_a R_a(m) \geq \mu_k R_k(m)$ and $\mu_a R_a(m+1) \leq \mu_k R_k(m+1)$. By dividing the two inequalities term by term and then applying (7), we obtain

$$\frac{\log(1+g_a P_m)}{\log(1+g_b P_m)} \geq \frac{\log(1+g_a P_{m+1})}{\log(1+g_b P_{m+1})}.$$  \hspace{1cm} (11)

As $P_m < P_{m+1}$, $f(x) \equiv \log(1+P_m x)/\log(1+P_{m+1} x)$ is a monotonically increasing function for $x > 0$ by Lemma 2. As (11) implies $f(g_a) \geq f(g_b)$, we get $g_a \geq g_b$, and, hence, $k^*(m) \geq k^*(m+1)$. This proves the “only if” part of the theorem.

To prove the “if” part, we show that it is possible to determine, for an arbitrary power-balanced deterministic policy, a normal vector $\mathbf{\mu}^*$ for which the optimal policy in (8) becomes the given policy. We consider a power-balanced policy having the property in (10). If there exists any user $k$ that is not selected by the policy, then we set $\mu_k^* = 0$, so that the corresponding optimal policy does not serve the user. So, we may consider only the case when all users are selected. If

3Theorem 3 addresses that it is desirable to allocate a higher power to a user with a weaker channel gain. It is a newly-derived result but is consistent with some other results considered on different system models: The optimal subchannel allocation scheme for multiuser OFDM systems considered in [13] allocates a subchannel with a higher channel gain to a user with a lower average channel gain under the assumption that frequency selectivity is identical among all users. The optimal transmission power allocation scheme for multiuser OFDM systems considered in [14] showed that the optimal allocation has a tendency of assigning more transmission power to a weaker user.
we define $m_k \equiv \min\{m|k^*(m) = k, \ m = 1, 2, \cdots, M\}$ for $k = 1, 2, \cdots, K$, then the relation

$$m_1 \geq m_2 \geq \cdots \geq m_K = 1$$

holds due to (10). Note that $k^*(m) = k$ for $m_k \leq m < m_{k - 1}$.

For the transmission rate $R_k(m)$ in (7), $R_k(m)/R_{k+1}(m)$ is a monotonically increasing function of $m$ due to Lemma 2. So we get the relation

$$\frac{R_k(1)}{R_{k+1}(1)} < \frac{R_k(2)}{R_{k+1}(2)} < \cdots < \frac{R_k(M)}{R_{k+1}(M)} < 1. \tag{13}$$

We first set $\mu^*_k$ to an arbitrary positive value and determine $\mu^*_2, \cdots, \mu^*_K$ by the following process: For any two users $k$ and $k + 1$, $k = 1, 2, \cdots, K - 1$, we choose the ratio $\mu^*_{k+1}/\mu^*_k$ such that

$$\frac{R_k(m_k - 1)}{R_{k+1}(m_k - 1)} < \frac{\mu^*_{k+1}}{\mu^*_k} < \frac{R_k(m_k)}{R_{k+1}(m_k)}. \tag{14}$$

Then we get

$$\mu^*_k R_k(m) < \mu^*_{k+1} R_{k+1}(m), \ m < m_k,$$

$$\mu^*_k R_k(m) > \mu^*_{k+1} R_{k+1}(m), \ m \geq m_k. \tag{15}$$

In view of (12), (15) reduces to

$$\mu^*_k R_k(m) > \mu^*_1 R_1(m) > \cdots > \mu^*_1 R_1(m),$$

for $m < m_k - 1$,

$$\mu^*_k R_k(m) > \mu^*_{k+1} R_{k+1}(m) > \cdots > \mu^*_K R_K(m),$$

for $m \geq m_k,$

which implies that $\arg \max_k \mu^*_k R_k(m) = k$ for $m_k \leq m < m_{k - 1}$. Therefore, the optimal policy in (8) for the normal vector $\mu^*$ becomes the given policy. This proves the “if” part of the theorem.

According to the theorem, we find that one-to-one relation exists between the Pareto optimal policy and the power-balanced policy. This relation is governed by the inequalities in (16), through which the boundaries, $m_k$, of a Pareto optimal policy generate the normal vector $\mu$ of the corresponding optimal policy and vice versa.

The optimality of a power-balanced policy may be explained pictorially by Fig. 1. This figure plots the optimal scheduling scheme in (8) for a two-user case with $g_1 < g_2$. The optimal scheduling will select user 2 in all the states if $\mu_1 \leq \mu_2$, so it suffices to consider only the case with $\mu_1 > \mu_2$.

Then, by the concavity of the power-rate relationship, the scheduling metric of user 2 is larger than that of user 1 until the transmission power grows to the intersection of the two curves. In this region, it is optimal to schedule user 2 for transmission and user 1 is optimal beyond that. This corresponds to the operation of a power-balanced policy, and we can observe that the threshold is determined by the ratio of the weighting factors $\mu_k$’s.

Theorem 3 implies that the optimal deterministic policy is power-balanced and a power-balanced deterministic policy is Pareto optimal, i.e., not dominated by any other policy (see footnote 1). Since a throughput vector of a non-deterministic policy can be represented by a convex combination of deterministic policies, we can conclude that the capacity region is the convex hull of the throughput vectors yielded by all the power-balanced deterministic policies and the origin. In other words, the boundary of the capacity region is piecewise linear with respect to the vertices yielded by power-balanced deterministic policies as depicted in Fig. 2. More rigorously,

$$C = \left\{ T \left| 0 \preceq T \preceq \sum_{i=1}^{h(M,K)} d_i T_i^{PB}, \ d_i \geq 0, \ \sum_{i=1}^{h(M,K)} d_i = 1 \right. \right\}, \tag{17}$$

where $T_i^{PB}$ denotes the throughput vector yielded by the $i$-th power-balanced deterministic policy (i.e., the vertices) and $h(M,K)$ denotes the number of power-balanced deterministic policies for the number of state $M$ and the number of users $K$.

**Theorem 4:** The number of power-balanced deterministic policies, $h(M,K)$, is $(M + K - 1)$.

**Proof:** We consider a $1 \times M$ vector that is obtained by taking the user index $M$ times with repetition and by sorting
it in descending order. Then, we can obtain a deterministic power-balanced policy by allocating a state \( m \) to the user who has the index of the \( m \)-th component of the sorted vector. Thus, the number of power-balanced deterministic policies, \( h(M, K) \), is equal to the number of different combinations that take \( K \) different user indices \( M \) times with repetition. It is well-known that the number of combinations with repetition is given by \( \binom{M+K-1}{K} \) and this proves the theorem.

**B. Non-deterministic policy**

We have studied the properties of the optimal deterministic policy in the previous subsection. In general, however, the boundary of the capacity region is composed of both deterministic and non-deterministic policies. Thus, in order to obtain the whole Pareto optimal operation points, it is also necessary to investigate how to determine the corresponding optimal non-deterministic policies. In this subsection, we examine the properties that the optimal non-deterministic policy possesses.

We first generalize the concept of power-balance defined in (10) to encompass the non-deterministic case as well: Let \( \mathbf{k}^*(m) \) denote the set of users served at state \( m \), i.e.,

\[
\mathbf{k}^*(m) \equiv \{ k \mid d_1^{(1)}(m) > 0, k = 1, 2, \ldots, K, \ i = 1, 2, \ldots, I \}.
\]

Then, a policy is called power-balanced if

\[
\min_{k_m \in \mathbf{k}^*(m)} k_m \geq \max_{k_{m+1} \in \mathbf{k}^*(m+1)} k_{m+1}
\]

for \( m = 1, 2, \ldots, M-1 \). This means that, in a power-balanced policy, users are selected such that the channel gain of a user at a state is higher than that of any user served at a higher-power state. Based on this generalized definition, we establish below that the optimal non-deterministic policy is also power-balanced.

**Theorem 5:** A Pareto optimal non-deterministic policy is power-balanced.

**Proof:** We prove the theorem by contradiction. Suppose a policy which is not power-balanced yields a Pareto optimal throughput vector \( \mathbf{T} \). Then there will exist two users \( a \) and \( b \) such that this policy serves user \( a \) with probability \( d_a \) at state \( m_a \) and serves user \( b \) with probability \( d_b \) at state \( m_b \), with \( m_a < m_b \) and \( g_a < g_b \). The throughputs of users \( a \) and \( b \) may be expressed by

\[
T_a = q_m d_a R_a(m_a) + \rho_a,
T_b = q_m d_b R_b(m_b) + \rho_b,
\]

in view of (4), where \( \rho_a \) and \( \rho_b \) denote the parts obtained at states other than \( m_a \) and \( m_b \), respectively. Now we modify the policy such that user \( b \) is served at state \( m_b \) and user \( a \) at state \( m_a \), resulting in the throughputs \( \tilde{T}_a \) and \( \tilde{T}_b \) with \( \tilde{T}_b = T_b \), and prove that \( \tilde{T}_a > T_a \) when \( q_m d_a R_a(m_a) \leq q_m d_b R_b(m_b) \) as well as when \( q_m d_a R_a(m_a) > q_m d_b R_b(m_b) \), as is detailed below. Then, it contradicts the Pareto optimal assumption above, which proves the theorem.

**Case 1)** When \( q_m d_a R_a(m_a) \leq q_m d_b R_b(m_b) \): Since \( q_m d_a R_a(m_a) - q_m d_a R_a(m_a) = q_m d_a R_a(m_b) \geq 0 \) for a value \( d_a \leq d_b \), the modified policy can allocate a \( (d_b - d_a) \) portion of state \( m_b \) to user \( a \), and can allocate a \( d_a \) portion of state \( m_b \) and a \( d_a \) portion of state \( m_a \) to user \( b \). Then, the resulting throughputs become

\[
T_a' = q_m d_a R_a(m_a) + q_m d_b R_b(m_a) + \rho_a, \quad T_b' = q_m d_a R_b(m_b) + q_m d_b R_b(m_b) + \rho_b = T_b.
\]

Thus, by (20) and (21), we get

\[
T_a' - T_a = q_m d_a R_b(m_a) \left\{ \frac{R_a(m_a)}{R_b(m_a)} - \frac{R_a(m_a)}{R_b(m_a)} \right\},
\]

which is positive by Lemma 2.

**Case 2)** When \( q_m d_a R_a(m_a) > q_m d_b R_b(m_b) \): For a value \( d_b \leq d_a \) with \( q_m d_a R_a(m_a) = q_m d_b R_a(m_b) \), the modified policy can allocate a \( (d_b - d_a) \) portion of state \( m_a \) and a \( d_a \) portion of state \( m_b \) to user \( a \), and can allocate a \( d_b \) portion of state \( m_a \) to user \( b \). Then, the resulting throughputs become

\[
T_a' = q_m d_a R_a(m_a) + q_m d_b R_a(m_b) + \rho_a, \quad T_b' = q_m d_a R_b(m_b) + \rho_b = T_b.
\]

In this case, by (20) and (23), we get

\[
T_a' - T_a = q_m d_a R_b(m_b) \left\{ \frac{R_a(m_b)}{R_b(m_b)} - \frac{R_a(m_b)}{R_b(m_b)} \right\},
\]

which is positive again by Lemma 2.

**V. TOTAL THROUGHPUT MAXIMIZATION**

Now that the optimality of the power-balance is established, we consider, for a total throughput maximization, how to allocate resources to each user in a power-balanced manner. Specifically, for a given requested throughput vector \( T^r = [T_1^r, T_2^r, \ldots, T_K^r] \), we consider the following problem (we call it Problem D):

\[
\max_{\mathbf{T}} \frac{1}{K} \sum_{k=1}^{K} T_k \quad \text{subject to } \mathbf{T} \in \mathcal{C} \text{ in } (4),
\]

\[
T_k \geq T_k^r \text{ for } k = 1, 2, \ldots, K.
\]

In order to solve the above problem, we establish the following lemma.

**Lemma 3:** A throughput vector \( \mathbf{T} \) is Pareto optimal if and only if it is a solution to Problem D for some \( T^r \).

**Proof:** First, we prove the “if” part by contradiction. Suppose that there exists a solution of Problem D, \( \mathbf{T} \), which is not Pareto optimal. Then, there will exist a feasible vector \( \mathbf{T}^* \) which dominates \( \mathbf{T} \), so we get the relations \( \sum_{k=1}^{K} T_k < \sum_{k=1}^{K} T_k^r \), and \( \mathbf{T} \preceq \mathbf{T}^* \). Therefore \( \mathbf{T}^* \) cannot be a solution of Problem D, which is a contradiction.

Now, we consider the “only if” part. Let \( \mathbf{T} \) be a Pareto optimal throughput vector. If we take \( T^r = \mathbf{T} \), then \( \mathbf{T} \) is the only throughput vector that satisfies the constraint in (25) and is thus the solution of Problem D.

Based on the property in Lemma 3, we develop an algorithm that determines the policy that yields the solution of Problem D. Let \( n_k \) be the smallest index of the state at which user \( k \) is served, i.e., \( n_k \equiv \min \{ m \mid k \in \mathbf{k}^*(m) \} \), and let \( t_k (0 < t_k \leq 1) \).
be the sum of the time-fraction of state \( n_k \) allocated to users 1, 2, \( \cdots \), \( k \), i.e., \( t_k = \sum_{i=1}^{I} \sum_{j=1}^{k} d_{i,j}^{(3)}(n_k) \).\(^5\) We define by \( \tau_k(n_k,t_k) \) the throughput that would be yielded if the states \( M, M - 1, \cdots, n_k + 1 \) and a \( t_k \) portion of state \( n_k \) were allocated to user \( k \), i.e.,

\[
\tau_k(n_k,t_k) = \sum_{i=n_k+1}^{M} q_i R_k(i) + q_{n_k} t_k R_k(n_k). \tag{26}
\]

By Theorem 5 and Lemma 3, it is established that the solution of Problem D can be yielded only by a power-balanced policy. In addition, since \( R_k(m) < R_{k+1}(m) \) by (7), we can easily show that the solution holds the relations \( T_k = T_k^{req} \) for \( k = 1, 2, \cdots, K - 1 \) and \( T_K \geq T_K^{req} \). In the following, we develop an algorithm that allocates resources maintaining these properties.

**Weakest User First Algorithm**

First, we consider the resource allocation to user 1, the weakest user. In order to maintain the power-balanced nature, we have no choice but to allocate the state with the highest power to user 1. Thus, we determine \( n_1 \) and \( t_1 \) by solving the equation

\[
\tau_1(n_1,t_1) = T_1^{req} \tag{27}
\]

such that the resulting throughput meets the condition \( T_1 = T_1^{req} \). Then, we allocate the states \( M, M - 1, \cdots, n_1 + 1 \) and a \( t_1 \) portion of state \( n_1 \) to user 1.

Once resources are allocated to users 1, 2, \( \cdots \), \( k - 1 \), in a power-balanced manner, then user \( k \) becomes the weakest user. So, to maintain power-balanced nature, we should allocate the highest available state to user \( k \), as above. Therefore, we determine \( n_k \) and \( t_k \) by solving the equation

\[
\tau_k(n_k,t_k) - \tau_k(n_{k-1},t_{k-1}) = T_k^{req}, \tag{28}
\]

such that the condition \( T_k = T_k^{req} \) is met, for \( k = 2, 3, \cdots, K - 1 \).\(^6\) If there exist no \( n_k \) and \( t_k \) that meet the relation in (28), we set the resulting throughput vector to 0 and terminate the process.\(^7\) Otherwise, we allocate the determined resources to each user as follows: If \( n_k = n_{k-1} \), we allocate a \((t_k - t_{k-1})\) portion of state \( n_k \) to user \( k \); otherwise, we allocate the states \( n_{k-1}, n_{k-1} - 1, n_{k-1} - 2, \cdots, n_k + 1 \) and a \((1 - t_{k-1})\) portion of state \( n_{k-1} \) and a \( t_k \) portion of state \( n_k \) to user \( k \).\(^8\)

After completing the above process sequentially for users 1, 2, \( \cdots \), \( K - 1 \), we allocate all the remaining states and time-fractions to user \( K \) to maximize the total throughput. The overall process can be arranged in a flow graph as shown in Fig. 3.

Fig. 4 shows an example of the time-fraction allocation of the above algorithm for the case \( n_1 = M - 1, n_2 = n_3 = M - 2, \cdots, n_{K-1} = 1 \). We named the algorithm a *weakest user first* (WUF) algorithm, as it was designed to allocate resources sequentially to the weakest user among the remaining users.

\(^5\)Note that the policy with \( t_k = 1 \) for all \( k \) is deterministic. In this case, \( n_k \) is equivalent to \( m_k \) in (12).

\(^6\)Note that the relation \( n_1 \geq n_2 \geq \cdots \geq n_{K-1} \) holds, which implies the power-balanced nature in view of (19).

\(^7\)This implies that the requested throughput vector \( T^{req} \) is not feasible.

\(^8\)In this sense, \( n_k \) and \( t_k \) may be said to represent the boundary of power state allocated to user \( k \).

As the WUF algorithm was developed to produce a power-balanced policy while holding the relations \( T_k = T_k^{req} \) for \( k = 1, 2, \cdots, K - 1 \) and \( T_K \geq T_K^{req} \), the resulting throughput vector solves Problem D, as long as the given \( T^{req} \) is feasible. Therefore, we may arrange the above discussion as a theorem as follows:

**Theorem 6:** The solution of Problem D is equivalent to the throughput vector \( T \) yielded by the WUF algorithm, as long as the requested throughput vector \( T^{req} \) is feasible.

Then, by relating Theorem 6 and Lemma 3, we can easily derive the following properties of the WUF algorithm.

**Corollary 1:** A throughput vector is Pareto optimal if and only if it is yielded by the WUF algorithm.

**Corollary 2:** A requested throughput vector \( T^{req} \) is feasible if and only if \( T^{req} \preceq T \) for the throughput vector yielded by the WUF algorithm \( T \).

The WUF algorithm has the advantage that it maximizes the total throughput under the constraints on each user’s throughput. Aside from such performance gain, it renders an
easy means to check the feasibility of the requested throughput vector.9

VI. EXTENSION TO FADING CHANNELS

So far, we have discussed the optimal policy for time-invariant channel gains. Now we extend the discussion to the case of fading channels where each user’s channel gain varies randomly in time. In support of this, we introduce the concept of channel state as follows: We assume that channel gain of each user is independent of those of the other users. Then, the channel gain of user \( k \) is a random variable expressed by

\[
G_k = S_k \bar{g}_k
\]

for a random variable \( S_k \) and a deterministic value \( \bar{g}_k \). Here, \( S_k \) is a random variable with \( E\{S_k\} = 1 \), representing the short term fading component of the channel gain. We denote by \( S_k \) the set of all the realizations of \( S_k \). The value \( \bar{g}_k \) is a user-specific deterministic variable of user \( k \) which determines the average channel gain of that user. We sort the user index in the ascending order of the average channel gain,

\[
0 < \bar{g}_1 < \bar{g}_2 < \cdots < \bar{g}_K
\]

Then, a random vector defined by \( S = \{S_1, S_2, \ldots, S_K\} \) represents the channel state of all the users at a given time instant. Also, we define a set \( \mathcal{A} \) as the set of all the possible realizations of \( S \), whose cardinality is given by \( |\mathcal{A}| = |S_1| \times |S_2| \times \cdots \times |S_K| \). In order to distinguish the state defined by the available transmission power, we state that the power state is in state \( m \) if the transmission power is given by \( P_m \). We denote by \( K^{*}(s,m) \) the user index to whom the power state \( m \) is allocated when the channel state is given by \( S = s = [s_1, s_2, \ldots, s_K] \) for an \( s \in \mathcal{A} \).

For the simplicity of discussion, we restrict in this section our consideration to the optimal deterministic policy with a non-increasing normal vector \( \mu \) in (8), \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \). This assumption is not unrealistic because it gives a larger weighting factor to a weaker user, which is quite natural for a fair resource allocation. We note that a normal vector with \( \mu_1 = \mu_2 = \cdots = \mu_K \) corresponds to the maximal rate scheduling that maximizes the total throughput without any fairness criterion.

We define by \( \mathcal{A}(k,m) \) the set of channel states where a user weaker than or equal to user \( k \) is selected at the power state \( m \). More specifically, we define

\[
\mathcal{A}(k,m) \equiv \{ s | K^{*}(s,m) \leq k, s \in \mathcal{A} \}.
\]

Then, with the above assumption, we have the following theorem which explains how the optimality of the power-balanced policy is reflected in fading channels.

**Theorem 7:** For an optimal deterministic policy with \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \), it holds

\[
\mathcal{A}(k,m) \supseteq \mathcal{A}(k,m-1)
\]

for \( k = 1, 2, \cdots, K \) and \( m = 2, 3, \cdots, M \).

9Since only the throughput of the strongest user is allowed to exceed the throughput requirement, a throughput vector obtained by using the WUF algorithm may appear unfair especially when each user’s throughput requirement \( T_{m}^{eq} \) is too small. This potential unfairness originates from the fact that each user’s requirement can be met with small amount of wireless resources. So, in such a case, one may adjust the requested throughput vector \( T^{eq} \) to a larger one and perform the WUF algorithm again.

**Proof:** We assume that the optimal policy selects a user stronger than user \( k \) at channel state \( s \) and power state \( m \). We can write it by \( k^{*}(s,m) = k^{*} > k \). Then, by (8), we get

\[
\mu_k \log_2(1 + s_k \bar{g}_k P_m) < \mu_{k^{*}} \log_2(1 + s_{k^{*}} \bar{g}_{k^{*}} P_m).
\]

Since \( \mu_k \geq \mu_{k^{*}} \), we have \( s_k \bar{g}_k < s_{k^{*}} \bar{g}_{k^{*}} \) from (32). Then, by applying \( P_m > P_{m-1} \) to Lemma 2, we have

\[
\frac{\mu_k \log_2(1 + s_k \bar{g}_k P_{m-1})}{\mu_{k^{*}} \log_2(1 + s_{k^{*}} \bar{g}_{k^{*}} P_{m-1})} < 1.
\]

This relation (33) implies that, for a given channel state, the user \( k^{*} \) who is selected at power state \( m \) has a larger scheduling metric at power state \( m - 1 \) as well. This means that user \( k \) cannot be selected by the optimal policy at power state \( m - 1 \), once the power state \( m \) is assigned to a stronger user \( k^{*} > k \). We can write this relation by

\[
s \in \mathcal{A} - \mathcal{A}(k,m) \Rightarrow s \in \mathcal{A} - \mathcal{A}(k,m-1).
\]

This means that, when the transmission power reduces from \( P_m \) to \( P_{m-1} \), a channel state not included in \( \mathcal{A}(k,m) \) is never assigned to a user weaker than or equal to user \( k \). This proves the theorem.

The above theorem implies that, in the optimal policy, the set of channel states assigned to weaker users shrinks as the available transmission power decreases. It also implies that, in order to achieve the optimality, a stronger power state should be allocated to a weaker user who has a lower average channel gain regardless of the channel state. In this sense, we can interpret Theorem 7 as the theorem that extends the optimality of the power-balanced policy to the cases of fading channels. From this theorem, we obtain the following corollary.

**Corollary 3:** For an optimal deterministic policy with \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \), it holds that

\[
Pr(k^{*}(M) = 1) \geq Pr(k^{*}(m) = 1)
\]

for \( m = 1, 2, \cdots, M - 1 \),

\[
Pr(k^{*}(1) = K) \geq Pr(k^{*}(m) = K)
\]

for \( m = 2, 3, \cdots, M \).

This corollary implies that the rate of scheduling for the weakest user, user 1, at power state \( M \) is higher than or equal to those at any other power states. In other words, if we compare the selection probability of the weakest user at different power states, that user is selected by an optimal policy most frequently at power state \( M \) which has the largest transmission power. In addition, the rate of scheduling for the strongest user, the user \( K \), at power state 1 is higher than or equal to those at any other power states.

VII. NUMERICAL EXAMPLES

In order to describe the capacity region explicitly, we present some numerical examples in this section. We consider the case of two users in different channel environments. We assume that the number of states, \( M \), is 20 and the probability of each state, \( q_m \), is 1/20. We also assume that the
transmission power is a quantized Gaussian random variable with the mean of 1 and the standard deviation of 1/3.\textsuperscript{10}

Fig. 5 depicts the capacity region obtained when the channel environment is $g_1 = 0$ dB and $g_2 = 10$ dB. Overlaid in the figure are the achievable regions of the round-robin and the counter-power-balanced policies. The round-robin policy serves the constituent users without considering the currently available amount of transmission power. In this case, the policy is expressed as a convex combination of the two deterministic policies that always serve a single user. The counter-power-balanced policy, in this figure, refers to the channel allocation policy that assigns a higher power state to a stronger user, in opposite to the power-balanced policy. We observe that the achievable regions of the two comparing policies are only a proper subset of the capacity region. Note that the circles at the boundary of the capacity region are the throughput vectors yielded by power-balanced deterministic policies and the lines connecting the circles are those yielded by power-balanced non-deterministic policies. We also observe that the counter-power-balanced policy renders even a smaller achievable region than the round-robin policy.

Fig. 6 depicts the capacity region obtained when the channel environment is changed to $g_1 = -3$ dB and $g_2 = 13$ dB. We observe that the performance difference between the optimal policy and the round-robin policy is increased when compared with the case of Fig. 5. This is attributed to the property that, due to the concavity of the throughput, it is more beneficial to allocate a higher-power state to a weaker user than to a stronger user. In general, as the difference between the user channel gains increases, the power-balanced policy can take more advantage of this property in resource allocation. For the same reason, the performance gain of the optimal policy also increases as the variance of the transmission power increases as shown in Fig. 7 which is plotted with the doubled standard deviation.

\textsuperscript{10}A quantized Gaussian random variable refers to the discrete random variable that is obtained by partitioning the range of a continuous Gaussian random variable into $M$ intervals and taking the middle of each interval as the realization having the probability of that interval.

Table I shows an example of the resulting channel allocation of the optimal policy for a two-user fading channel case. Each user’s channel randomly varies in a set $S_1 = S_2 = \{0.6, 0.8, 1.0, 1.2, 1.4\}$, so there are 25 channel states in this system. We set $g_1 = -3$ dB, $g_2 = 13$ dB, $\mu_1 = 8$, and $\mu_2 = 1$. This table represents the smallest index of the power state at which user 1 is selected at each channel state. For example, the index at the channel state $[0.8 \ 1.0]$, 15, implies that this channel state is assigned to user 1 when the power state is larger than or equal to 15. The term “none” implies that the channel state associated with it is never assigned to user 1 regardless of the power state. In this table, we observe that the set of the channel states allocated to user 1 shrinks as the power state $m$ decreases. For example, all the channel states except $[0.6 \ 1.4]$ are allocated to user 1 when the power state is 20 but the three channel states, $[0.6 \ 1.2]$, $[0.6 \ 1.0]$, and $[0.6 \ 0.8]$, are dropped off when the power state decreases to 19. This demonstrates Theorem 7 established in the previous section.
TABLE I
Channel allocation of the optimal policy for a two-user fading channel case ($S_1 = S_2 = \{0.6, 0.8, 1.0, 1.2, 1.4\}$, $\bar{g}_1 = -3$ dB, $\bar{g}_2 = 13$ dB, $\mu_1 = 8$, and $\mu_2 = 1$.)

<table>
<thead>
<tr>
<th>$S_2 \setminus S_1$</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>18</td>
<td>9</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0.8</td>
<td>20</td>
<td>12</td>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1.0</td>
<td>20</td>
<td>15</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>1.2</td>
<td>20</td>
<td>17</td>
<td>10</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>1.4</td>
<td>None</td>
<td>18</td>
<td>12</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

VIII. Conclusions

In this paper, we have studied the capacity region of a multi-user shared channel with time-varying transmission power. We first showed that the capacity region is the convex hull of multiple deterministic policies, each of which serves a single user at each state. Then, we have shown that a power-balanced policy which serves a weaker user at a higher transmission power state is equivalent to a Pareto optimal policy. This property gives the insight that it is advantageous to allocate a better resource having a higher transmission power to a user with weaker channel gain. Moreover, the optimality of the power-balanced policy is maintained regardless of fairness criterion among the constituent users. Based on this property, we developed the weakest user first (WUF) algorithm that renders an easy means to maximize the total throughput and to examine the feasibility of the requested throughput vector. In addition, we have showed that the optimality of the power-balanced policy can be extended to the fading channels.

Theorems established in this paper are noteworthy in that they establish for the first time the equivalence of Pareto optimality and power-balanced nature in resource allocation with time-varying transmission power. The theorems verified that one-to-one relation exists between the Pareto optimal and the power-balanced policies: For any desired optimal policy prescribed by a particular fairness criterion, there always exists a corresponding power-balanced policy whose power state boundaries are determined according to the given fairness criterion. In fact, the WUF algorithm itself describes the process of determining the boundaries, which can be applied to other resource allocation related optimization problems as well. By virtue of the structural simplicity of the WUF algorithm, the optimal resource allocation strategy we have discussed in this paper can be effectively applied to various wireless access systems with time-varying power, especially to the IS-2000 1xEV-DV and 3GPP HSDPA systems.

REFERENCES


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